Binomial Formula:

$$\binom{n}{0}=\binom{n}{n}=1,\ \binom{n}{1}=\binom{n}{n-1}=n,\quad \binom{n}{k}=\binom{n}{n-k}=\frac{n!}{k!(n-k)!}\qquad n,k\in\mathbb{N}$$

$$(x+y)^n=\binom{n}{0}x^0y^n+\binom{n}{1}x^1y^{n-1}+\binom{n}{2}x^2y^{n-2}+\cdots$$

$$+\binom{n}{n-1}x^{n-1}y^1+\binom{n}{n}x^ny^0=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

(1) number of terms is n+1

(2) General Term $=\binom{n}{k}x^{k}y^{n-k}$

3) Middle term: n is even and k= $\frac{n}{2}+1$, hence

Middle term= $\binom{n}{\frac{n}{2}}x^{\frac{n}{2}}y^{\frac{n}{2}}$

(4) Two middle terms: n is odd and k= $\frac{n+1}{2}$, or k= $\frac{n+3}{2}$

hence two middle terms when n is odd are:

$$\binom{n}{\frac{n+1}{2}}x^{\frac{n+1}{2}}y^{n-\frac{n+1}{2}}=\binom{n}{\frac{n+1}{2}}x^{\frac{n+1}{2}}y^{\frac{n-1}{2}}$$ and

$$\binom{n}{\frac{n+3}{2}}x^{\frac{n+3}{2}}y^{n-\frac{n+3}{2}}=\binom{n}{\frac{n+3}{2}}x^{\frac{n+3}{2}}y^{\frac{n-3}{2}}$$

Exercise. **Expand** $\left(2x^{2}-3y^{3}\right)^{5}$

Solution.

$$\binom{5}{0}=\binom{5}{5}=1,\quad \binom{5}{1}=\binom{5}{4}=5,\quad \binom{5}{2}=\binom{5}{5-2}=\frac{5!}{2!(5-2)!}=\frac{3!\times4\times5}{2!\times3!}=\frac{20}{2}=10 \qquad n,k\in\mathbb{N}$$

$$\Rightarrow\left(2x^{2}-3y^{3}\right)^{5}=\binom{5}{0}(2x^{2})^{0}(-3y^{3})^{5}+\binom{5}{1}(2x^{2})^{1}(-3y^{3})^{4}$$

$$+\binom{5}{2}(2x^{2})^{2}(-3y^{3})^{3}+\binom{5}{3}(2x^{2})^{3}(-3y^{3})^{2}+\binom{5}{4}(2x^{2})^{4}(-3y^{3})^{1}$$

$$+\binom{5}{5}(2x^{2})^{5}(-3y^{3})^{0}=(-3y^{3})^{5}+5(2x^{2})^{1}(-3y^{3})^{4}+10(2x^{2})^{2}(-3y^{3})^{3}$$

$$+10(2x^{2})^{3}(-3y^{3})^{2}+5(2x^{2})^{4}(-3y^{3})^{1}+(2x^{2})^{5}$$

$$=-243y^{15}+810x^{2}y^{12}-1080x^{4}y^{9}+720x^{6}y^{6}-240x^{8}y^{3}+32x^{10}$$

Exercise. **Find the middle term of** $\left(4x^{2}+2y^{5}\right)^{8}$

Solution.

$$\text{Middle term}=\binom{n}{\frac{n}{2}}(4x^{2})^{\frac{n}{2}}(2y^{5})^{\frac{n}{2}}=\binom{8}{\frac{8}{2}}(4x^{2})^{\frac{8}{2}}(2y^{5})^{\frac{8}{2}}$$

$$=\frac{8!}{4!(8-4)!}4^{4}x^{8}\times 2^{4}y^{20}=\frac{4!\times 5\times 6\times 7\times 8}{4!\times 4!}8^{4}x^{8}y^{20}$$

$$=140\times 8^{4}x^{8}y^{20}$$

Constant Term. **To find the constant term of an expansion find and simplify the General Term of the expansion, then the exponent of the variable should be zero.**

Exercise. **Find the constant term of** $\left(4x^{6}-\frac{\sqrt{2}}{x^{4}}\right)^{15}$

Solution.

$$General\ \text{Term}=\binom{15}{k}(4x^{6})^{k}\left(-\frac{\sqrt{2}}{x^{4}}\right)^{15-k}=\binom{15}{k}4^{k}x^{6k}\frac{(-\sqrt{2})^{15-k}}{(x^{4})^{15-k}}$$

$$=\binom{15}{k}4^{k}x^{6k}\frac{(-\sqrt{2})^{15-k}}{x^{60-4k}}=\binom{15}{k}4^{k}(-\sqrt{2})^{15-k}x^{6k-60+4k}=\binom{15}{k}4^{k}(-\sqrt{2})^{15-k}x^{10k-60}$$

$$10k-60=0\Rightarrow 10k=60\Rightarrow k=6$$

$$k=6\Rightarrow \text{Constant Term}=\binom{15}{6}4^{6}(-\sqrt{2})^{15-6}x^{0}=\binom{15}{6}4^{6}(-\sqrt{2})^{15-6}$$

$$=\binom{15}{6}4^{6}(-\sqrt{2})^{9}=-\binom{15}{6}4^{6}\sqrt{2}(\sqrt{2})^{8}=-\binom{15}{6}4^{6}\sqrt{2}(2^{4})=-\binom{15}{6}16\times 4^{6}\sqrt{2}$$