

Vector Spaces

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By **addition** we mean a rule for associating with each pair of objects \mathbf{u} and \mathbf{v} in V an object $\mathbf{u} + \mathbf{v}$, called the **sum** of \mathbf{u} and \mathbf{v} ; by **scalar multiplication** we mean a rule for associating with each scalar k and each object \mathbf{u} in V an object $k\mathbf{u}$, called the **scalar multiple** of \mathbf{u} by k . If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m , then we call V a **vector space** and we call the objects in V **vectors**.

1. If \mathbf{u} and \mathbf{v} are objects in V , then $\mathbf{u} + \mathbf{v}$ is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in V , called a **zero vector** for V , such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all \mathbf{u} in V .
5. For each \mathbf{u} in V , there is an object $-\mathbf{u}$ in V , called a **negative** of \mathbf{u} , such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and \mathbf{u} is any object in V , then $k\mathbf{u}$ is in V .
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

Example. \mathbb{R}^n Is a Vector Space

Let $V = \mathbb{R}^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n -tuples; that is,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) \\ k\mathbf{u} &= (ku_1, ku_2, \dots, ku_n)\end{aligned}$$

The set $V = \mathbb{R}^n$ is closed under addition and scalar multiplication because the foregoing operations produce n -tuples as their end result, and these operations satisfy Axioms 2, 3, 4, 5, 7, 8, 9, and 10

Example. The Vector Space of Infinite Sequences of Real Numbers

Let V consist of objects of the form

$$\mathbf{u} = (u_1, u_2, \dots, u_n, \dots)$$

in which $u_1, u_2, \dots, u_n, \dots$ is an infinite sequence of real numbers. We define two infinite sequences to be *equal* if their corresponding components are equal, and we define addition and scalar multiplication componentwise by

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n, \dots) + (v_1, v_2, \dots, v_n, \dots) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n, \dots) \\ k\mathbf{u} &= (ku_1, ku_2, \dots, ku_n, \dots)\end{aligned}$$

We leave it as an exercise to confirm that V with these operations is a vector space. We will denote this vector space by the symbol \mathbb{R}^{∞} .

Example. A Vector Space of 2×2 Matrices

Let V be the set of 2×2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \\ k\mathbf{u} &= k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}\end{aligned}\tag{1}$$

The set V is closed under addition and scalar multiplication because the foregoing operations produce 2×2 matrices as the end result. Thus, it remains to confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold. Some of these are standard properties of matrix operations. For example, Axiom 2 follows from Theorem 1.4.1a since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

To confirm that Axiom 4 is satisfied, we must find a 2×2 matrix $\mathbf{0}$ in V for which $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$ for all 2×2 matrices in V . We can do this by taking

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With this definition,

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

and similarly $\mathbf{u} + \mathbf{0} = \mathbf{u}$. To verify that Axiom 5 holds we must show that each object \mathbf{u} in V has a negative $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ and $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. This can be done by defining the negative of \mathbf{u} to be

$$-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

With this definition,

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and similarly $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. Finally, Axiom 10 holds because

$$1\mathbf{u} = 1 \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

Example. The Vector Space of Real-Valued Functions

Let V be the set of real-valued functions that are defined at each x in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in V and if k is any scalar, then define the operations of addition and scalar multiplication by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) \quad (2)$$

$$(k\mathbf{f})(x) = kf(x) \quad (3)$$

Example. A Set That Is Not a Vector Space

Let $V = \mathbb{R}^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$, and $k = 7$, then

$$\mathbf{u} + \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$$

$$k\mathbf{u} = 7\mathbf{u} = (7 \cdot 2, 0) = (14, 0)$$

The addition operation is the standard one from \mathbb{R}^2 , but the scalar multiplication is not. In the exercises we will ask you to show that the first nine vector space axioms are satisfied. However, Axiom 10 fails to hold for certain vectors. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1 \cdot u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus, V is not a vector space with the stated operations.