Gram-Schmidt Process

Gram-Schmidth Process

Definition. A set of two or more vectors in an inner product space is said to be **orthogonal** if all pairs in the set are orthogonal.

If each vector in the set has norm 1, we say the set is **orthonormal.**

Example. In $\mathbb{R}^3$, the sets

$$A = \{(1,0,0),(0,1,0),(0,0,1)\}$$ and

$$B = \{(2,-2,1),(3,2,-2),(2,7,10)\}$$

are orthogonal. Set *A* is orthonormal, but set *B is not!*

Solution.

Let $V_1 = (1,0,0),\ \ V_2 = (0,1,0),\ \ V_3 = (0,0,1).$

Evidently $V_1 \bullet V_2 = 0,\ \ V_1 \bullet V_3 = 0\ \ and\ \ V_2 \bullet V_3 = 0.$

Also $\|V_1\| = 1\ \ and\ \ \|V_2\| = 1\ \ and\ \ \|V_3\| = 1.$

So $A = \{(1,0,0),(0,1,0),(0,0,1)\}$ is orthonormal.

Now let $X_1 = (2, -2, 1), \quad X_2 = (3, 2, -2), \quad X_3 = (2, 7, 10).$

$$X_1 \bullet X_2 = (2, -2, 1) \bullet (3, 2, -2) = 2 \times 3 - 2 \times 2 + 1 \times (-2) = 6 - 4 - 2 = 0.$$

$$X_1 \bullet X_3 = (2, -2, 1) \bullet (2, 7, 10) = 2 \times 2 + (-2) \times 7 + 1 \times 10 = 4 - 14 + 10 = 0.$$

$$X_2 \bullet X_3 = (3, 2, -2) \bullet (2, 7, 10) = 3 \times 2 + 2 \times 7 + (-2) \times 10 = 6 - 14 - 20 = 0.$$

So $B = \{(2, -2, 1), (3, 2, -2), (2, 7, 10)\}$ is orthogonal.

$\| (2, -2, 1) \| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{5} \neq 1,$

thus B is not orthonormal.

Homework.

(a) For what values of *a* and *b, the set*

$$C = \{(\tfrac{1}{2}, a, b), (\tfrac{-1}{2}, b, a)\}$$

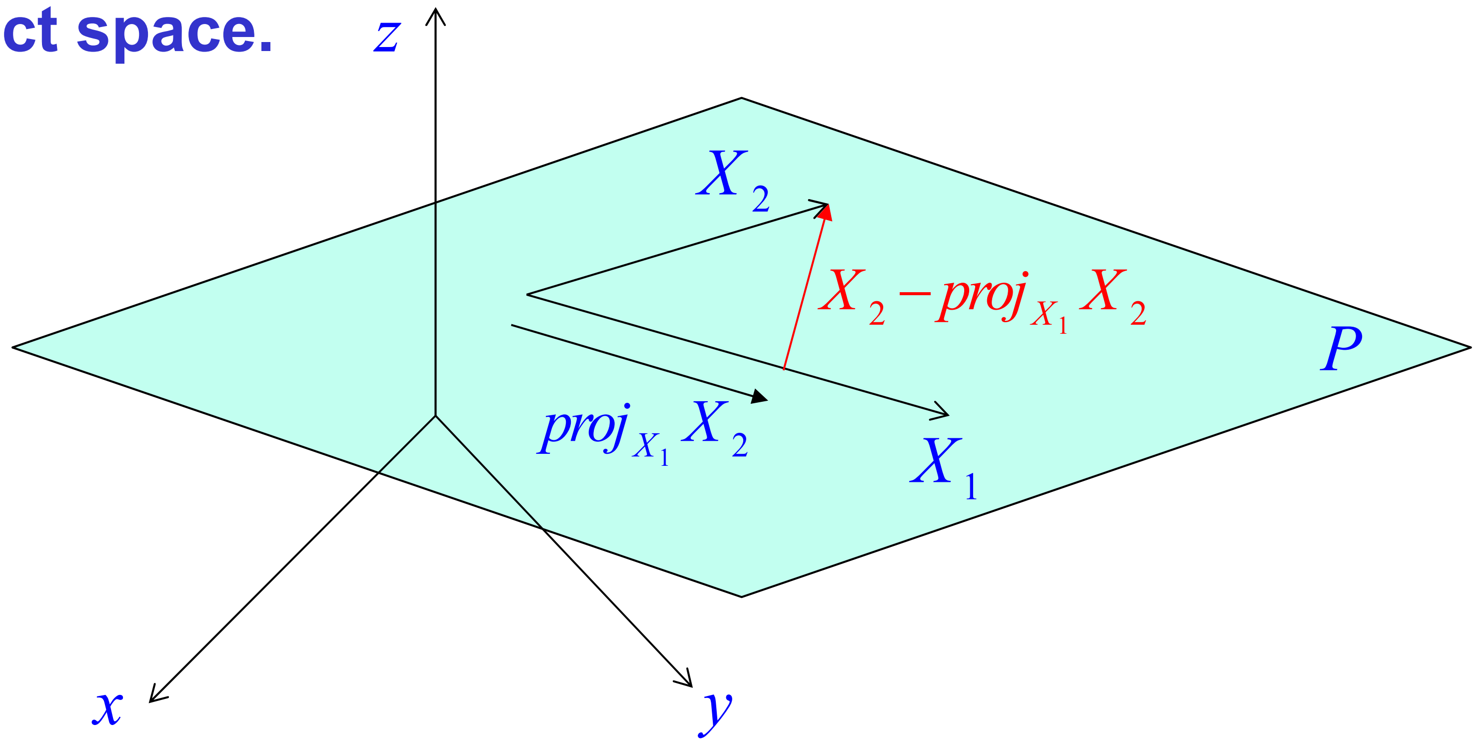
is an orthonormal set in $\mathbb{R}^3$.

(b) Expand *C* to an orthonormal basis for $\mathbb{R}^3$.

Gram-Schmidth Process

Gram-Schmidth Process is a method for finding an orthogonal (or orthonormal) basis for a subspace of an inner product space.

Idea!



So the set $\{X_1, X_2 - proj_{X_1} X_2\} = \{X_1, X_2 - \frac{X_2 \bullet X_1}{X_1 \bullet X_1} X_1\}$

is an orthogonal basis for the plane P.

Gram-Schmidth Process. Let $\{X_1, X_2, X_3, \ldots, X_p\}$ be a basis for a subspace ***W*** of an inner produce space.

To find an orhogonal (or orthonormal) basis for ***W*** :

Step 1. Let $V_1 = X_1$.

Step 2. $$V_2 = X_2 - \frac{\langle X_2, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1$$

Indeed if $W_2 = Span\{V_1\}$, **then**

$$V_2 = X_2 - proj_{W_1} X_2$$

Step 3. $V_3 = X_3 - \frac{\langle X_3, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 - \frac{\langle X_3, V_2 \rangle}{\langle V_2, V_2 \rangle} V_2$

Indeed if $W_2 = Span\{V_1, V_2\}$, **then**

$$V_3 = X_3 - proj_{W_2} X_3$$

Step 4.

$$V_4 = X_4 - \frac{\langle X_4, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 - \frac{\langle X_4, V_2 \rangle}{\langle V_2, V_2 \rangle} V_2 - \frac{\langle X_4, V_3 \rangle}{\langle V_2, V_3 \rangle} V_3$$

means if $W_3 = Span\{V_1, V_2, V_3\}$ **,then**

$$V_4 = X_4 - proj_{W_3} X_4$$

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Step *p*.

$$V_p = X_p - \frac{\langle X_p, V_1\rangle}{\langle V_1, V_1\rangle}V_1 - \frac{\langle X_p, V_2\rangle}{\langle V_2, V_2\rangle}V_2 - \ldots - \frac{\langle X_p, V_{p-1}\rangle}{\langle V_{p-1}, V_{p-1}\rangle}V_{p-1}$$

means if $W_{p-1} = Span\{V_1, V_2, \ldots, V_{p-1}\}$, **then**

$$V_p = X_p - proj_{W_{p-1}} X_p$$

Step *. After finding V_i in each of the above steps, replace V_i by a suitable scalar of V_i to simplify the calculation.

Then $\{V_1, V_2, V_3, \ldots, V_p\}$ is an orthogonal basis for ***W***,

and $\left\{\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|}, \ldots, \frac{V_p}{\|V_p\|}\right\}$ is an orthonormal basis for ***W***.

Example. (a) Show that $\{(1,1,1,1),(0,1,1,1),(0,0,1,1)\}$

is a linearly independent set in $\mathbb{R}^4$.

(b) Find an orthonormal basis for

$$W = Span\{(1,1,1,1),(0,1,1,1),(0,0,1,1)\}.$$

Solution. (a)

$$\text{Let } X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{Consider A}=[X_1, X_2, X_3] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now use row operations on matrix A to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

As A has 3 pivot columns, X_1, X_2 and X_3 *are linearly independent*.

Solution. (b)

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Step 1. $V_1 = X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$

Step 2. $V_2 = X_2 - \dfrac{X_2 \bullet V_1}{V_1 \bullet V_1} V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \dfrac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

Step *. We replace V_2 **by** $4V_2$, so we consider $V_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$

Step 3. $$V_3 = X_3 - \frac{X_3 \bullet V_1}{V_1 \bullet V_1} V_1 - \frac{X_3 \bullet V_2}{V_2 \bullet V_2} V_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\overset{1}{\cancel{2}}}{\underset{2}{\cancel{4}}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\overset{1}{\cancel{2}}}{\underset{6}{\cancel{12}}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} + \frac{3}{6} \\ \frac{-1}{2} - \frac{1}{6} \\ 1 - \frac{1}{2} - \frac{1}{6} \\ 1 - \frac{1}{2} - \frac{1}{6} \end{bmatrix}.$$

Step *. We replace V_3 by

$6V_3$, so consider $$V_3 = \begin{bmatrix} -3+3 \\ -3-1 \\ 6-3-1 \\ 6-3-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \\ 2 \end{bmatrix}.$$

Thus $\{V_1, V_2, V_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 2 \\ 2 \end{bmatrix} \right\}$

is an orthogonal basis for W .

and so $\left\{ \dfrac{V_1}{\|V_1\|}, \dfrac{V_2}{\|V_2\|}, \dfrac{V_3}{\|V_3\|} \right\} = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \end{bmatrix}, \begin{bmatrix} 0 \\ -\sqrt{\frac{2}{3}} \\ \frac{6}{\sqrt{6}} \\ \frac{6}{\sqrt{6}} \end{bmatrix} \right\}$

is an orthonormal basis for W .

Homework.

Let $W = Span\left\{\begin{bmatrix}-1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-2\\3\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\6\\4\end{bmatrix}\right\}.$

Find an orthonormal basis for W .

Example.

Consider P_2 (the vector space of the polynomials of degree ≤ 2) with the inner product

$$<p,q>=\int_{-1}^{1} p(x).q(x)\,dx$$

Find an orthonormal basis for P_2.

Solution. We know tha $\{1, x, x^2\}$ is a basis for P_2.

Let $X_1 = 1$, $X_2 = x$, and $X_3 = x^2$.

Step 1. $V_1 = X_1 = 1$.

Step 2. $V_2 = X_2 - \frac{\langle X_2, V_1\rangle}{\langle V_1, V_1\rangle} V_1 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1$

$$\langle x, 1\rangle = \int_{-1}^{1} x\,.1\,dx = 0$$

$$\langle 1,1\rangle = \int_{-1}^{1} 1.1\,dx = 1(1+1) = 2 \quad \Rightarrow V_2 = x - \frac{0}{2}1 = x$$

Step 3. $V_3 = X_3 - \frac{\langle X_3, V_1\rangle}{\langle V_1, V_1\rangle} V_1 - \frac{\langle X_3, V_2\rangle}{\langle V_2, V_2\rangle} V_2 = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1,1\rangle} 1 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle} x$

$$\langle x^2, 1\rangle = \int_{-1}^{1} x^2.1\,dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

$$\langle x^2, x\rangle = \int_{-1}^{1} x^2.x\,dx = \int_{-1}^{1} x^3\,dx = 0$$

$$V_3 = x^2 - \frac{\frac{2}{3}}{2} - \frac{0}{\frac{2}{3}} x = x^2 - \frac{1}{3}$$

Thus $\{V_1, V_2, V_3\} = \left\{1, x, x^2 - \frac{1}{3}\right\}$ is an orthogonal basis for P_2.

Now $\|V_1\|^2 = \|1\|^2 = \langle 1,1\rangle = 2 \Rightarrow \|1\| = \sqrt{2}$.

$$\|V_2\|^2 = \|x\|^2 = \langle x, x\rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big]_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$$

$$\|V_3\|^2 = \|x^2 - \frac{1}{3}\|^2 = \langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx = \frac{x^5}{5} - \frac{2}{9}x^3 + \frac{1}{9}x\Big]_{-1}^{1} = \frac{8}{45}$$

Thus $\left\{ \frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|} \right\} = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$

is an orthonormal basis for P_2.