Determinants

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE

(a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$

(b) $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$ The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

EXAMPLE

Find the determinant of the matrix

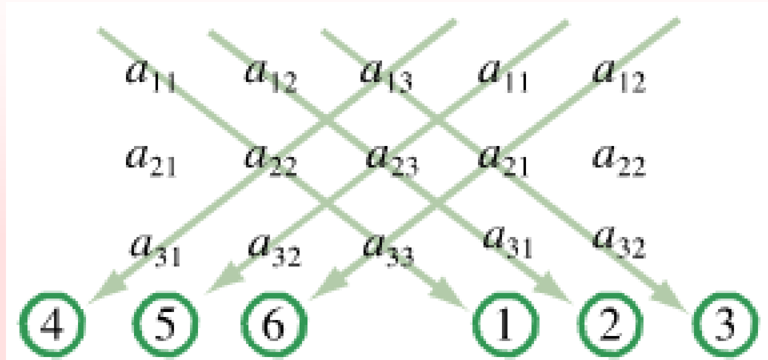
$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

EXAMPLE A Technique for Evaluating 3 × 3 Determinants



$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{matrix}\right| \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

EXAMPLE Compute

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{vmatrix}$$

$$-4\begin{vmatrix} 0 & 4 & 4 \\ -1 & 1 & 0 \\ 2 & 4 & 1 \end{vmatrix}$$

$$+2\begin{vmatrix} 0 & 1 & 4 \\ -1 & 0 & 0 \\ 2 & 0 & 1 \end{vmatrix}$$

$$-3\begin{vmatrix} 0 & 1 & 4 \\ -1 & 0 & 1 \\ 2 & 0 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix}$$

EXAMPLE

$$\begin{vmatrix} -1 & 0 & 0 & 4 \\ -2 & 0 & -1 & 1 \\ 3 & 2 & 0 & 4 \\ 0 & -3 & 1 & -2 \end{vmatrix} = (-1)\begin{vmatrix} 0 & -1 & 1 \\ 2 & 0 & 4 \\ -3 & 1 & -2 \end{vmatrix} - (0)\begin{vmatrix} -2 & -1 & 1 \\ 3 & 0 & 4 \\ 0 & 1 & -2 \end{vmatrix} + (0)\begin{vmatrix} -2 & 0 & 1 \\ 3 & 2 & 4 \\ 0 & -3 & -2 \end{vmatrix} - (4)\begin{vmatrix} -2 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 0 & -1 & 1 \\ 2 & 0 & 4 \\ -3 & 1 & -2 \end{vmatrix} - 4\begin{vmatrix} -2 & 0 & -1 \\ 3 & 2 & 0 \\ 0 & -3 & 1 \end{vmatrix}$$

THEOREM

If a matrix A of order n is upper triangular, lower triangular, or diagonal, then $\det A = a_{11}a_{22} \cdots a_{nn}$, the product of the entries on the main diagonal.

EXAMPLE

Compute $\det A$, where

$$A = \begin{bmatrix} 2 & 4 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Solution By Theorem 4, $\det A = (2)(1)(3)(-2) = -12$.

Row Operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

EXAMPLE Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

SOLUTION Add 2 times row 1 to row 3 to obtain

$$\det A = \det \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

because the second and third rows of the second matrix are equal.

EXAMPLE Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.

SOLUTION A good way to begin is to use the 2 in column 1 as a pivot, eliminating the -2 below it. Then use a cofactor expansion to reduce the size of the determinant, followed by another row replacement operation. Thus

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

An interchange of rows 2 and 3 would produce a "triangular determinant." Another approach is to make a cofactor expansion down the first column:

$$\det A = (-2)(1) \begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} = -2 \cdot (15) = -30$$

■

EXAMPLE Compute det A, where

$$A = \begin{bmatrix} \frac{1}{3} & 0 & \frac{3}{4} \\ \frac{2}{5} & -1 & \frac{3}{2} \\ \frac{1}{8} & -\frac{3}{4} & \frac{5}{4} \end{bmatrix}.$$

Solution Since 12 clears the first row, 10 the second, and 8 the last, we multiply the determinant by $\frac{1}{12}$ and the first row by 12, and proceed similarly with the other rows.

$$\det A = \left(\frac{1}{12}\right)\left(\frac{1}{10}\right)\left(\frac{1}{8}\right) \det \begin{bmatrix} 4 & 0 & 9 \\ 4 & -10 & 15 \\ 1 & -6 & 10 \end{bmatrix}$$

Now -2 can be factored out of the second column:

$$\det A = \left(\frac{1}{12}\right)\left(\frac{1}{10}\right)\left(\frac{1}{8}\right)(-2) \det \begin{bmatrix} 4 & 0 & 9 \\ 4 & 5 & 15 \\ 1 & 3 & 10 \end{bmatrix}$$

The final value is $\det A = (-1/480)(83) = -83/480$. ■

EXAMPLE Find det A by using the process described in the preceding computational note, where

$$A = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix}.$$

Solution By interchanging the second and third rows, and then adding multiples of each row to the bottom row, we obtain the sequence of equations

$$\det A = -\det \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 1 & 3 & 4 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 5 & 3 \end{bmatrix} = -\det \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & \frac{11}{2} \end{bmatrix}.$$

Finally, $\det A = -(1)(-1)(2)(11/2) = 11$. ■

A square matrix A is invertible if and only if $\det A \neq 0$.

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

EXAMPLE

Determine whether or not the matrix A is invertible, where A is given by

$$A = \begin{bmatrix} 2 & 1 & -3 & 1 \\ -3 & -2 & 0 & 2 \\ 2 & 1 & 0 & -1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Solution $\det A = -8 \neq 0$, and thus A is invertible.