Trigonometric Substitutions For Integrals

Use the following trigonometric substitutions

$a^2 - u^2$	$u = a.\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\theta = \sin^{-1}\frac{u}{a}$
$a^2 + u^2$	$u = a.\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\theta = \tan^{-1}\frac{u}{a}$
$u^2 - a^2$	$u = a.\sec\theta$ $0 \le \theta < \frac{\pi}{2} \vee \pi \le \theta < \frac{3\pi}{2}$	$\theta = \sec^{-1}\frac{u}{a}, u \ge a$ $\theta = 2\pi - \sec^{-1}\frac{u}{a}, u < -a$

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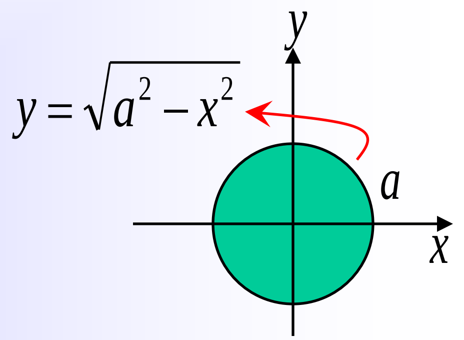
Exercise.

Show that the area of a circle of radius a is πa^2

Solution.

$$A = 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$x = a.\sin\theta$$



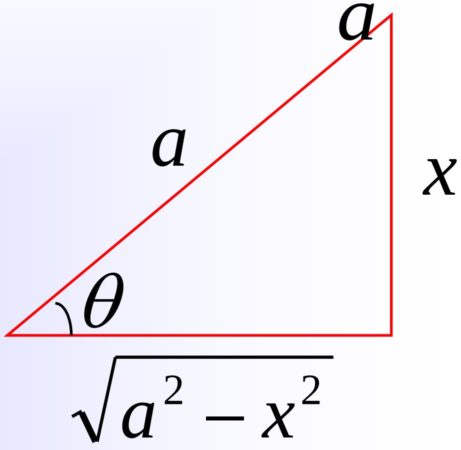
$$\int \sqrt{a^2 - x^2}\,dx \;= \int \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta d\theta$$

$$= \int a|\cos\theta| \times a\cos\theta d\theta = \int a^2\cos^2\theta d\theta$$

$$= a^2\int \cos^2\theta d\theta = a^2\int \frac{1+\cos 2\theta}{2}d\theta = \frac{a^2}{2}\int (1+\cos 2\theta)d\theta$$

$$=\frac{a^2}{2}\left(\theta+\frac{Sin2\theta}{2}\right) \quad =\frac{a^2}{2}(\theta+\sin\theta.\cos\theta)$$

$$\sin\theta=\frac{x}{a}$$



$$=\frac{a^2}{2}\left(\sin^{-1}\frac{x}{a}+\frac{x}{a}\times\frac{\sqrt{a^2-x^2}}{a}\right)$$

$$\cos\theta=\frac{\sqrt{a^2-x^2}}{a}$$

$$A=4\times\frac{a^2}{2}\left(\sin^{-1}\frac{x}{a}+\frac{x.\sqrt{a^2-x^2}}{a^2}\right)\Bigg|_0^a$$

$$=2a^2\left(\frac{\pi}{2}+0-0\right) \quad =\pi a^2$$

Exercise. $$I=\int\frac{e^{t}dt}{(e^{2t}+8e^{2t}+7)}=?$$

Solution.

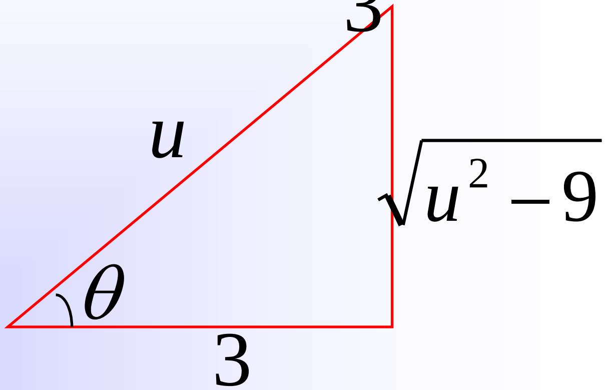
$$e^{t}+4=u\Rightarrow e^{t}dt=du \qquad u=3\sec\theta\Rightarrow du=3\sec\theta\tan\theta d\theta$$

$$I=\int\frac{e^{t}dt}{((e^{t}+4)^{2}-9)^{\frac{3}{2}}}=\int\frac{du}{(u^{2}-9)^{\frac{3}{2}}}=\int\frac{3\sec\theta\tan\theta d\theta}{(9\sec^{2}\theta-9)^{\frac{3}{2}}}$$

$$=\int\frac{3\sec\theta\tan\theta d\theta}{27\tan^{3}\theta}=\frac{1}{9}\int\frac{\sec\theta}{\tan^{2}\theta}d\theta=\frac{1}{9}\int\frac{\frac{1}{\cos\theta}}{\frac{\sin^{2}\theta}{\cos^{2}\theta}}d\theta$$

$$=\frac{1}{9}\int\frac{\cos\theta}{\sin^{2}\theta}d\theta \quad =\frac{1}{9}\int\sin^{-2}\theta\cos\theta d\theta$$

$$\sec\theta=\frac{u}{3}$$



$$=-\frac{1}{9}(\sin\theta)^{-1}+c=\frac{-1}{9}\csc\theta+c$$

$$\csc\theta=\frac{u}{\sqrt{u^{2}-9}}$$

$$=\frac{-1}{9}\frac{u}{\sqrt{u^{2}-9}}+c=\frac{-1}{9}\frac{e^{t}+4}{\sqrt{(e^{t}+4)^{2}-9}}+c$$

Exercise.

Find the length of the arc $y = Ln\ x, \quad 1 \le x \le 3.$

Solution.

$$l = \int_1^3 \sqrt{1+(\frac{1}{x})^2}\,dx = \int_1^3 \sqrt{\frac{1+x^2}{x^2}}\,dx = \int_1^3 \frac{1}{x}\sqrt{1+x^2}\,dx$$

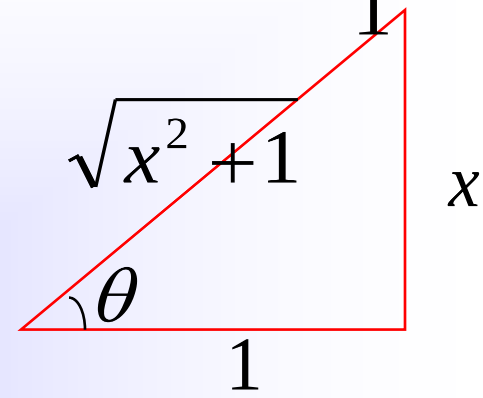
$$x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$$

$$\Rightarrow \int \frac{1}{x}\sqrt{1+x^2}\,dx = \int \frac{1}{\tan\theta}\sqrt{1+\tan^2\theta}\,\sec^2\theta\, d\theta$$

$$= \int \frac{\sec^3\theta}{\tan\theta}\,d\theta = \int \frac{d\theta}{\sin\theta\cos^2\theta}$$

$$= \int \frac{\sin^2\theta + \cos^2\theta}{\sin\theta.\cos^2\theta} d\theta \quad = \int \frac{\sin\theta}{\cos^2\theta} d\theta + \int \frac{d\theta}{\sin\theta}$$

$$\tan\theta = \frac{x}{1}$$



$$= \int \sin\theta.\cos^{-2}\theta d\theta + \int \csc\theta d\theta$$

$$= \frac{1}{\cos\theta} + \ln\left|\csc\theta - \cot\theta\right|$$

$$\frac{1}{\cos\theta} = \frac{\sqrt{x^2+1}}{1}$$

$$= \sqrt{x^2+1} + \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right|$$

$$\csc\theta = \frac{\sqrt{x^2+1}}{x}$$

$$\Rightarrow I = \sqrt{x^2+1} + \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right|\Bigg]_1^3$$

$$\cot\theta = \frac{1}{x}$$