Exercises. **Find the Maclaurin series for the following functions.**

1) $y=\dfrac{1}{1-x}$

2) $y=\dfrac{1}{x+1}$

3) $y=\dfrac{1}{1+x^{2}}$

4) $y=\dfrac{1}{(1-x)^{2}}$

5) $y=\ln(1-x)$

6) $y=\ln(1+x)$

7) $y=\tan^{-1}x$

8) $y=\dfrac{1}{(1+x)^{2}}$

9) $y=\ln(\dfrac{1+x}{1-x})$

10) $y=\dfrac{\tan^{-1}x}{x}$

Solution. 1) $y=\frac{1}{1-x}=\frac{(1-x)+(x-x^2)+(x^2-x^3)+(x^3-x^4)+\ldots}{1-x}$

$$=\frac{(1-x)+x(1-x)+x^2(1-x)+x^3(1-x)+\ldots}{1-x}$$

$$=\frac{(1-x)\left(1+x+x^2+x^3+\ldots\right)}{1-x}$$

$$=1+x+x^2+x^3+\ldots \qquad |x|<1$$

$$\Rightarrow y=\frac{1}{1-x}=\sum_{n=0}^{+\infty}x^n, \qquad |x|<1 \qquad (1)$$

$$(2)\, y = \frac{1}{1+x}$$

$$x \xrightarrow{(1)} -x \Rightarrow \frac{1}{1+x} = \sum_{k=0}^{+\infty} (-x)^k = \sum_{k=0}^{+\infty} (-1)^k x^k, |-x| < 1 \Rightarrow |x| < 1$$

$$(3)\, y = \frac{1}{1+x^2}$$

$$x \xrightarrow{(2)} x^2 \Rightarrow \frac{1}{1+x^2} = \sum_{k=0}^{+\infty} (-1)^k \left(x^2\right)^k = \sum_{k=0}^{+\infty} (-1)^k x^{2k}, |x| < 1$$

$$(4)\, \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots = \sum_{k=1}^{+\infty} kx^{k-1}, |x| < 1$$

$$(5)\ \ln(1-x)=\ln|x-1|=\int_0^x \frac{1}{1-t}\,dt=\sum_{k=0}^{+\infty}\frac{x^{k+1}}{k+1} \qquad |x|<1$$

$$(6)\ \ln(1+x)=\int_0^x \frac{1}{1+t}\,dt=\sum_{k=0}^{+\infty}(-1)^k\,\frac{x^{k+1}}{k+1} \qquad |x|<1$$

$$(7)\ \tan^{-1}x=\int_0^x \frac{1}{1+t^2}\,dt=\sum_{k=0}^{+\infty}(-1)^k\,\frac{x^{2k+1}}{2k+1} \qquad |x|<1$$

$$(8)\ \frac{1}{(1+x)^2}=-\left(\frac{1}{1+x}\right)'=-\left(-1+2x-3x^2+\ldots+(-1)^k kx^{k-1}+\ldots\right)=$$

$$-\sum_{k=1}^{+\infty}(-1)^k kx^{k-1}=\sum_{k=1}^{+\infty}k(-1)^{k+1}x^{k-1}, \quad |x|<1$$

$$(9)\ln\left(\frac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)=$$

$$(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-...)+(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+..)$$

$$=2(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}+..)=2\sum_{k=0}^{+\infty}\frac{x^{2k+1}}{2k+1}$$

$$(10)\frac{\tan^{-1}x}{x}=\frac{1}{x}\tan^{-1}x. \quad \text{Now by}\,(7)$$

$$=\frac{1}{x}\sum_{k=0}^{+\infty}(-1)^k\ \frac{x^{2k+1}}{2k+1}=\sum_{k=0}^{+\infty}(-1)^k\ \frac{x^{2k}}{2k+1}$$

Exercise. **Prove that:** $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots)$

Solution. **Put** $x = 1$ **in**

$$(7)\ \tan^{-1} x = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$\Rightarrow \tan^{-1} 1 = \frac{\pi}{4} = (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$$

$$\Rightarrow \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots)$$

Taylor and Maclaurin Series

Taylor Series of y=f(x) about x=a is $$\sum_{n=0}^{+\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor series about x=0 is called the Maclaurin Series. So

Maclaurin Series of y=f(x) is $$\sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Exercise. **Find the Maclaurin series of the following functions**

1) $f(x)=e^x$, 2) $f(x)=x^3e^x$

3) $f(x)=\sin x$, 4) $f(x)=\cos x$, 5) $f(x)=\cos^2 x$

Solution.

1) $f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f''(x) = e^x \Rightarrow f''(0) = 1 \qquad \Rightarrow \dots \Rightarrow f^{(n)}(0) = 1$

$\Rightarrow$ **Maclaurin Series of** $= \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$

2) $f(x) = x^3 e^x \Rightarrow$ **Maclaurin Series of** $= x^3 \sum_{n=0}^{+\infty} \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^{n+3}}{n!}$

$$(3)\ f(x)=\sin x \to f(0)=0 \qquad f'(x)=\cos x \to f'(0)=1$$

$$f''(x)=-\sin x \to f''(0)=0 \to f''(x)=-\cos x \to f''(0)=-1$$

$$f^{(4)}(x)=\sin x \Rightarrow f^{(4)}(0)=0$$

$$=\sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^{n}=0+\frac{1}{1!}x^{1}+0-\frac{1}{3!}x^{3}+0+\frac{1}{5!}x^{5}+\ldots$$

$\Rightarrow$ **Maclaurin Series of** $=\sum_{n=0}^{+\infty}\frac{(-1)^{n}}{(2n+1)!}x^{2n+1}$

4) $f(x)=\cos x$

Maclaurin Series of $\sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \Rightarrow$

Now take the derivative on both sides:

$\Rightarrow$ **Maclaurin Series of** $\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$$5)\ f(x)=\cos^2 x \qquad f(x)=\cos^2 x=\frac{1+\cos 2x}{2}=\frac{1}{2}(1+\cos 2x)\Rightarrow$$

$\Rightarrow$ **Maclaurin Series of**

$$\cos^2 x=\frac{1}{2}\left(1+\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n}\right)$$

$$=\frac{1}{2}\left(1+\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}2^{2n}x^{2n}\right)$$

$$=\frac{1}{2}+\sum_{n=0}^{+\infty}\frac{(-1)^n 2^{2n-1}}{(2n)!}x^{2n}$$

Exercise. **Find the Taylor series of the functions about the given point.**

1) $f(x) = \sqrt[3]{x}, \quad a = 1$

2) $f(x) = \sin x, a = 3, a = -2,$

3) $f(x) = Lnx, \quad a \quad arbitrary$

4) $f(x) = x^3 e^x$

Solution.

$$1)\ f(x)=\sqrt[3]{x} \qquad a=1$$

$$f(x)=x^{\frac{1}{3}} \Rightarrow f(1)=1 \qquad f'(x)=\frac{1}{3}x^{-\frac{2}{3}} \Rightarrow f'(1)=\frac{1}{3}$$

$$f''(x)=-\frac{1}{3}\times\frac{2}{3}\times x^{-\frac{5}{3}} \Rightarrow f''(1)=-\frac{1}{3}\times\frac{2}{3}=-\frac{1\times 2}{3^2}$$

$$f'''(x)=\frac{1}{3}\times\frac{2}{3}\times\frac{5}{3}\times x^{\frac{-8}{3}} \Rightarrow f'''(1)=\frac{1}{3}\times\frac{2}{3}\times\frac{5}{3}=\frac{1\times 2\times 5}{3^3}$$

$$f^{(4)}(x)=-\frac{1}{3}\times\frac{2}{3}\times\frac{5}{3}\times\frac{8}{3}\times x^{-\frac{11}{3}}$$

$$\Rightarrow f^{(4)}(1) = -\frac{1}{3}\times\frac{2}{3}\times\frac{5}{3}\times\frac{8}{3} = -\frac{1\times 2\times 5\times 8}{3^4}$$

$$\Rightarrow \ldots \Rightarrow f^{(n)}(1) = \frac{(-1)^{n+1}\times 2\times 5\times 8\times \ldots \times(3n-4)}{3^n}$$

$\Rightarrow$ Taylor series about $a=1$ is

$$= \sum_{n=0}^{+\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n = 1+\frac{1}{3}(x-1)+\sum_{n=2}^{+\infty}\frac{(-1)^{n+1}\times 2\times 5\times \ldots \times(3n-4)}{3^n\times n!}(x-1)^n$$

$$2)\quad f(x) = \sin x, \quad a = 3$$

Solution.

$$\Rightarrow f(x) = \sin\left(\left(x-3\right)+3\right) = \cos 3.\sin\left(x-3\right) + \sin 3.\cos\left(x-3\right)$$

Now it is enough to substitute x by x-3 in the Maclaurin series of

$$\sin x \quad \text{and} \quad \cos x$$

So Taylor series $= \cos 3. \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!}(x-3)^{2n+1}$

$$+\sin 3. = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!}(x-3)^{2n}$$

3) $f(x) = Lnx$ about arbitrary a

Solution. $Lnx = Ln(a + x - a) = Ln(a(1 + \frac{x-a}{a}))$

$$= Lna + Ln(1 + \frac{x-a}{a})$$

Now it is enough to substitute x by $\left(\frac{x-a}{a}\right)$ in

$$\ln(1+x) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

So Taylor series=

$$\ln a + \sum_{k=0}^{+\infty} (-1)^k \frac{(\frac{x-a}{a})^{k+1}}{k+1} = \ln a + \sum_{k=0}^{+\infty} \frac{(-1)^k}{a^{k+1}.(k+1)} (x-a)^{k+1}$$

$$(4)\ f(x)=x^3e^x \quad about \quad a=-2$$

Solution.

$$x^3e^x=((x+2)-2)^3e^{(x+2)-2}=$$

$$\left((x+2)^3-6(x+2)^2+12(x+2)-8\right)e^{-2}.e^{x+2}=$$

$$e^{-2}(x+2)^3e^{x+2}-6e^{-2}(x+2)^2e^{x+2}+12e^{-2}(x+2)e^{x+2}-8e^{-2}e^{x+2}$$

$$e^x=\sum_{n=0}^{+\infty}\frac{x^n}{n!}\Rightarrow e^{x+2}=\sum_{n=0}^{+\infty}\frac{(x+2)^n}{n!}\Rightarrow$$

$$e^{-2}(x+2)^3 e^{x+2} - 6e^{-2}(x+2)^2 e^{x+2} + 12e^{-2}(x+2)e^{x+2} - 8e^{-2}e^{x+2}$$

$$= e^{-2}(x+2)^3 \sum_{n=0}^{+\infty} \frac{(x+2)^n}{n!} - 6e^{-2}(x+2)^2 \sum_{n=0}^{+\infty} \frac{(x+2)^n}{n!}$$

$$+12e^{-2}(x+2)\sum_{n=0}^{+\infty} \frac{(x+2)^n}{n!} - 8e^{-2} \sum_{n=0}^{+\infty} \frac{(x+2)^n}{n!}$$

$$= \sum_{n=0}^{+\infty} \frac{e^{-2}(x+2)^{n+3}}{n!} + \sum_{n=0}^{+\infty} \frac{-6e^{-2}(x+2)^{n+2}}{n!}$$

$$+ \sum_{n=0}^{+\infty} \frac{12e^{-2}(x+2)^{n+1}}{n!} + \sum_{n=0}^{+\infty} \frac{-8e^{-2}(x+2)^{n}}{n!}$$

$$=\sum_{n=3}^{+\infty}\frac{e^{-2}\left(x+2\right)^{n}}{(n-3)!}+\sum_{n=2}^{+\infty}\frac{-6e^{-2}\left(x+2\right)^{n}}{(n-2)!}$$

$$+\sum_{n=1}^{+\infty}\frac{12e^{-2}\left(x+2\right)^{n}}{(n-1)!}+\sum_{n=0}^{+\infty}\frac{-8e^{-2}\left(x+2\right)^{n}}{n!}=$$

$$=\left(-8e^{-2}-8e^{-2}\left(x+2\right)+\frac{-8e^{-2}\left(x+2\right)^{2}}{2!}\right)+$$

$$\left(12e^{-2}\left(x+2\right)+12e^{-2}\left(x+2\right)^{2}\right)+\left(-6e^{-2}\left(x+2\right)^{2}\right)$$

$$\sum_{n=3}^{+\infty}\left(\frac{e^{-2}}{(n-3)!}-\frac{6e^{-2}}{(n-2)!}+\frac{12e^{-2}}{(n-1)!}-\frac{8e^{-2}}{n!}\right)\left(x+2\right)^{n}$$

$$= -8e^{-2} + (-8e^{-2} + 12e^{-2})(x+2) +$$

$$(\frac{-8e^{-2}}{2!} + 12e^{-2} - 6e^{-2})(x+2)^2 +$$

$$\sum_{n=3}^{+\infty}\left(\frac{1}{(n-3)!} - \frac{6}{(n-2)!} + \frac{12}{(n-1)!} - \frac{8}{n!}\right)e^{-2}(x+2)^n$$

$$= -8e^{-2} + 4e^{-2}(x+2) + 2e^{-2}(x+2)^2 +$$

$$\sum_{n=3}^{+\infty}\left(\frac{1}{(n-3)!} - \frac{6}{(n-2)!} + \frac{12}{(n-1)!} - \frac{8}{n!}\right)e^{-2}(x+2)^n$$