Infinite Series

Telescoping Series

$$\sum_{k=n}^{m}(a_k - a_{k-1}) = a_m - a_{n-1}$$

$$\sum_{k=n}^{m}(a_k - a_{k+1}) = a_n - a_{m+1}$$

Exercises. **Find the convergent values of the convergent series.**

$$1-\sum_{n=3}^{+\infty}\frac{1}{n(n-1)}$$

$$2-\sum_{n=1}^{+\infty}\frac{1}{25n^2-15n-4}$$

$$3-\sum_{n=1}^{+\infty}Ln\left(1+\frac{1}{n}\right)$$

$$4-\sum_{n=2}^{+\infty}\frac{\sqrt{n^2-1}-n}{\sqrt{n^2-n}}$$

$$5-\sum_{n=5}^{+\infty}\frac{2n+3}{(n+1)^4+2(n+1)^3+(n+1)^2}$$

Solutions.

$$1.\ \sum_{n=3}^{+\infty}\frac{1}{n(n-1)}$$

$$S_n=\sum_{k=3}^{n}\frac{1}{k(k-1)}=\sum_{k=3}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)=\frac{-1}{n}+\frac{1}{2}$$

$$\Rightarrow \lim_{n\to+\infty}S_n=1/2$$

So the series converges to $\frac{1}{2}$

$$\sum_{n=1}^{+\infty} \frac{1}{25n^2 - 15n - 4} = ?$$

$$s_n = \sum_{k=1}^{n} \frac{1}{25k^2 + 5k - 20k - 4}$$

$$= \sum_{k=1}^{n} \frac{1}{5k(5k+1) - 4(5k+1)} = \sum_{k=1}^{n} \frac{1}{(5k+1)(5k-4)}$$

$$= \sum_{k=1}^{n} \frac{1}{5}\left(\frac{1}{(5k-4)} - \frac{1}{(5k+1)} \right) = \frac{1}{5}\sum_{k=1}^{n}\left(\frac{1}{5(k-1)+1} - \frac{1}{5k+1} \right)$$

$$= \frac{1}{5}\left(\frac{1}{5(1-1)+1} - \frac{1}{5n+1} \right) = \frac{1}{5}\left(1 - \frac{1}{5n+1} \right) \Rightarrow \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{1}{5}\left(1 - \frac{1}{5n+1} \right) = \frac{1}{5}$$

$$3-\sum_{n=1}^{+\infty} Ln\left(1+\frac{1}{n}\right)$$

$$\Rightarrow s_n = \sum_{k=1}^{n} Ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n} Ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n}\left(Ln\left(k+1\right)-Ln\left(k\right)\right)$$

$$= Ln\left(n+1\right)-Ln\left(1\right) = Ln\left(n+1\right)$$

$$\Rightarrow \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} Ln\left(n+1\right) = +\infty$$

$$4-\sum_{n=2}^{+\infty}\frac{\sqrt{n^2-1}-n}{\sqrt{n^2-n}}$$

$$s_n=\sum_{k=2}^{n}\frac{\sqrt{k-1}.\sqrt{k+1}-\sqrt{k}.\sqrt{k}}{\sqrt{k-1}.\sqrt{k}}=\sum_{n=2}^{n}(\sqrt{\frac{k+1}{k}}-\sqrt{\frac{k}{k-1}})$$

$$=\sum_{k=2}^{n}(\sqrt{\frac{k+1}{k}}-\sqrt{\frac{k}{k-1}})=\sqrt{\frac{n+1}{n}}-\sqrt{\frac{2}{2-1}}$$

$$\Rightarrow \lim_{n\to+\infty} s_n=\sqrt{1}-\sqrt{2}=1-\sqrt{2}$$

$$5-\sum_{n=5}^{+\infty}\frac{2n+3}{(n+1)^4+2(n+1)^3+(n+1)^2}=\sum_{n=5}^{+\infty}\frac{2n+3}{(n+1)^2((n+1)^2+2(n+1)+1)}$$

$$\Rightarrow s_n=\sum_{k=5}^{n}\frac{2k+3}{(k+1)^2((k+2)^2)}=\sum_{k=5}^{n}\left(\frac{1}{(k+1)^2}-\frac{1}{(k+2)^2}\right)$$

$$=\sum_{k=5}^{n}\left(\frac{1}{(k+1)^2}-\frac{1}{(k+2)^2}\right)=\frac{1}{(5+1)^2}-\frac{1}{(n+2)^2}$$

$$\Rightarrow \lim_{n\to+\infty}s_n=\frac{1}{(5+1)^2}=\frac{1}{36}$$

Geometric Series

Each series of the form $\sum_{n=k_0}^{+\infty} ar^n = ar^{k_0} + ar^{k_0+1} + \ldots \quad a \neq 0$

is called a geometric series.

r is called the ratio and ar^{k_0} **is the first term of the series.**

Theorem **In a geometric series**

(a) If $|r| < 1$ **then the series converges to** $\dfrac{\text{First Term}}{\text{1- ratio}}$

(b) If $|r| \geq 1$ **then the series is divergent.**

Exercises. **Find the convergent values of the convergent series.**

1) $\displaystyle\sum_{n=5}^{\infty}\frac{2^{n-1}}{3^{n}}$

2) $\displaystyle\sum_{n=0}^{+\infty}\frac{5^{n}-2^{n}}{10^{n}}$

3) $\displaystyle\sum_{n=1}^{+\infty}3^{n-1}\times 2^{1-2n}$

4) $\displaystyle\sum_{n=1}^{+\infty}\frac{2+(-1)^{n}}{2^{n}}$

5) $\displaystyle\sum_{n=3}^{+\infty}\frac{1^{n}+2^{n}+3^{n}}{4^{n}}$

6) $\displaystyle\sum_{k=4}^{\infty}\frac{2^{k+1}-3^{k+4}}{5^{k+2}}$

Solutions.

$$1)\ \sum_{n=5}^{\infty}\frac{2^{n-1}}{3^n} = \sum_{n=5}^{\infty}\frac{2^n}{3\times 3^n} = \tfrac{1}{2}\sum_{n=5}^{\infty}\left(\frac{2}{3}\right)^n$$

$$= \frac{1}{2}\left(\frac{\left(\frac{2}{3}\right)^5}{1-\frac{2}{3}}\right) = \frac{16}{81}$$

$$2)\ \sum_{n=0}^{+\infty}\frac{5^n-2^n}{10^n} = \sum_{n=0}^{+\infty}\left(\frac{5}{10}\right)^n - \left(\frac{2}{10}\right)^n = \frac{1}{\tfrac{1}{2}} - \frac{1}{\tfrac{1}{4}} = 2-\frac{5}{4}$$

$$3)\sum_{n=1}^{+\infty} 3^{n-1}\times 2^{1-2n} = \sum_{n=1}^{+\infty} \frac{3^{n-1}\times\left(\frac{1}{4}\right)^{n-1}}{\left(\frac{1}{4}\right)^{-\frac{1}{2}}} = \sum_{n=1}^{+\infty} \frac{3^{n-1}\times\left(\frac{1}{4}\right)^{n-1}}{2}$$

$$= \frac{1}{2}\sum_{n=1}^{+\infty}\left(\frac{3}{4}\right)^{n-1} = \frac{1}{2}\left(\frac{1}{\frac{1}{4}}\right) = 2$$

$$4)\sum_{n=1}^{+\infty} \frac{2+(-1)^n}{2^n} = \sum_{n=1}^{+\infty}\left(\frac{1}{2}\right)^{n-1} + \sum_{n=1}^{+\infty}\left(\frac{-1}{2}\right)^{n} = 2 - \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$