Integration of Rational Functions

Note. **Case1:**

$$\int \frac{dx}{(ax+b)^m} = \frac{1}{a}\int a(ax+b)^{-m}\,dx = \begin{cases} \frac{1}{a}\,\frac{(ax+b)^{-m+1}}{-m+1} & m \neq 1 \\ \frac{1}{a} Ln\,|\,ax+b\,| + c & m = 1 \end{cases}$$

$$\int \frac{du}{u^m} = \begin{cases} \frac{u^{-m+1}}{-m+1} + c & m \neq 1 \\ Ln\,|\,u\,| + c & m = 1 \end{cases}$$

Note. Case 2: $\int \frac{dx}{(ax^2+bx+c)^m}$ where $b^2-4ac<0$,

use completing the square to write $ax^2+bx+c=u^2+k^2, where\ k\in\mathbb{R}$.

Now for $\int \frac{dx}{(u^2+k^2)^m}$, use $u=k\tan\theta$

Exercise. $\int \frac{dx}{(x^2+x+1)^2}$

Exercise. $I = \int \frac{dx}{(x^2 + x + 1)^2}$

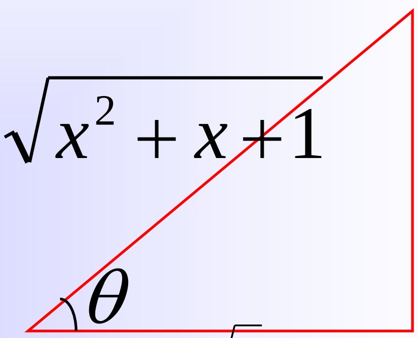
Solution. $\text{I} = \int \frac{dx}{\left((x + \frac{1}{2})^2 + \frac{3}{4}\right)^2}$

$x + \frac{1}{2} = u; \quad u = \frac{\sqrt{3}}{2}\tan\theta \quad \Rightarrow du = \frac{\sqrt{3}}{2}\sec^2\theta d\theta$

$$II = \int \frac{du}{(u^2 + \frac{3}{4})^2} = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{9}{16}(\sec^4\theta)} d\theta = \frac{8\sqrt{3}}{9}\int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{4\sqrt{3}}{9}\int(\cos 2\theta + 1)d\theta = \frac{4\sqrt{3}}{9}\left(\frac{\sin 2\theta}{2} + \theta\right) + c = \frac{4\sqrt{3}}{9}(\sin\theta\cos\theta + \theta) + c$$

$$\tan\theta = \frac{x + \frac{1}{2}}{\frac{3}{2}}$$



$$\sin\theta.\cos\theta = \frac{(x + \frac{1}{2})(\frac{\sqrt{3}}{2})}{x^2 + x + 1}$$

$$I = \frac{4\sqrt{3}}{9}\left(\frac{(x + \frac{1}{2})(\frac{\sqrt{3}}{2})}{x^2 + x + 1} + \tan^{-1}\frac{2(x + \frac{1}{2})}{\sqrt{3}}\right) + c$$

Note. Case 3: $\displaystyle\int \frac{a'x+e}{(ax^2+bx+c)^m}dx$, where $b^2-4ac<0$,

$$I=\int \frac{a'x+e}{(ax^2+bx+c)^m}dx=\int \frac{\frac{a'}{2a}(2ax+b)-\frac{a'b}{2a}+e}{(ax^2+bx+c)^m}dx$$

$$=\frac{a'}{2a}\int \frac{2ax+b}{(\underbrace{ax^2+bx+c}_{=u})^m}dx+\left(-\frac{a'b}{2a}+e\right)\int \frac{dx}{(ax^2+bx+c)^m}$$

Case 1 (first integral, with $u = ax^2+bx+c$) **Case 2** (second integral)

Exercise. $\displaystyle II=\int \frac{x+3}{(x^2+x+1)^2}dx$

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Solution.

$$II = \int \frac{\frac{1}{2}(2x+1)+\frac{5}{2}}{(x^2+x+1)^2}dx \qquad = \frac{1}{2}\int \frac{2x+1}{(x^2+x+1)^2}dx + \int \frac{\frac{5}{2}dx}{(x^2+x+1)^2}$$

$$= \frac{-1}{2(x^2+x+1)} + \frac{5}{2}\int \frac{dx}{(x^2+x+1)^2}$$

and we calculated $\int \frac{dx}{(x^2+x+1)^2}$ in the previous example.

Note. Case 4: $\int \frac{r(x)}{g(x)}dx \quad where$ degree $r(x) <$ degree $g(x)$,

Factor g(x) as the products of the polynomials of the form

$(ax^2+bx+c)^n, \ and \ (dx+e)^m, where \ b^2-4ac<0.$

Then split the fraction $\frac{r(x)}{g(x)}$ as follows:

$$\frac{r(x)}{(ax+b)^m(cx^2+dx+e)^n} =$$

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \ldots + \frac{A_m}{(ax+b)^m}$$

$$+ \frac{B_1(2cx+d)+C_1}{(cx^2+dx+e)} + \frac{B_2(2cx+d)+C_2}{(cx^2+dx+e)^2} + \frac{B_3(2cx+d)+C_3}{(cx^2+dx+e)^3}$$

$$+ \ldots + \frac{B_n(2cx+d)+C_n}{(cx^2+dx+e)^n}$$

To find A_i, B_i and C_i's, multiply both sides of the above equation by $(ax+b)^m(cx^2+dx+e)^n$.

Examples for splitting a fraction:

$$\frac{2x+1}{(3x+5)^2(4x-7)^4(x^2+x+1)^3}$$

$$=\frac{A}{(3x+5)}+\frac{B}{(3x+5)^2}+\frac{C}{(4x-7)}+\frac{D}{(4x-7)^2}+\frac{E}{(4x-7)^3}+\frac{F}{(4x-7)^4}$$

$$+\frac{G(2x+1)+H}{(x^2+x+1)}+\frac{I(2x+1)+J}{(x^2+x+1)^2}+\frac{K(2x+1)+L}{(x^2+x+1)^3}$$

$$\frac{4x^2+x+1}{(3x^2+2x+5)^2(x^2-x+1)^3}=\frac{A(6x+2)+B}{(3x^2+2x+5)}+\frac{C(6x+2)+D}{(3x^2+2x+5)^2}+$$

$$\frac{E(2x-1)+F}{(x^2-x+1)}+\frac{G(2x-1)+H}{(x^2-x+1)^2}+\frac{I(2x-1)+J}{(x^2-x+1)^3}$$

Exercise. $$\int \frac{dz}{3z^2+8z-3}$$

Solution.

$$=\int \frac{dz}{3z^2+8z-3}=\int \frac{dz}{(z+3)(3z-1)}$$

$$\frac{1}{(z+3)(3z-1)}=\frac{A}{z+3}+\frac{B}{3z-1} \qquad (1)$$

$$\Rightarrow 1=A(3z-1)+B(z+3) \qquad (2)$$

$$z=-3 \Rightarrow 1=-10A \Rightarrow A=-\frac{1}{10} \qquad z=\frac{1}{3} \Rightarrow 1=\frac{10}{3}B \Rightarrow B=\frac{3}{10}$$

$$\Rightarrow \int \frac{dz}{(3z-1)(z+3)} = \int \left(\frac{\frac{-1}{10}}{z+3} + \frac{\frac{3}{10}}{3z-1}\right)dz = \frac{-1}{10}\int \frac{dz}{z+3} + \frac{1}{10}\int \frac{3dz}{3z-1}$$

$$= \frac{-1}{10} Ln\left|z+3\right| + \frac{1}{10} Ln\left|3z-1\right| + c = \frac{1}{10} Ln\left|\frac{3z-1}{z+3}\right| + c$$

Exercise.

$$I = \int \frac{2x^2 - x + 2}{x^5 + 2x^3 + x} dx$$

Solution.

$$\frac{2x^2 - x + 2}{x^5 + 2x^3 + x} = \frac{2x^2 - x + 2}{x(x^2+1)^2} = \frac{A}{x} + \frac{B(2x)+C}{x^2+1} + \frac{D(2x)+E}{(x^2+1)^2}$$

$$\frac{2x^2 - x + 2}{x(x^2+1)^2} = \frac{A(x^2+1)^2 + (B(2x)+C)x(x^2+1) + (D(2x)+E)x}{x(x^2+1)^2}$$

$$\Rightarrow A(x^2+1)^2 + (B(2x)+C)x(x^2+1) + (D(2x)+E)x = 2x^2 - x + 2$$

$$x = i \Rightarrow (2iD + E)i = -2 - i + 2 \Longrightarrow Ei - 2D = -i \Rightarrow D = 0, E = -1$$

$$x = 0 \Rightarrow A = 2$$

$$x=1 \Rightarrow 4A+2(2B+C)+(2D+E)=3 \Rightarrow 8+2(2B+C)-1=3 \Rightarrow 2B+C=-2$$

$$x=-1 \Rightarrow 4A-2(-2B+C)-(-2D+E)=5 \Rightarrow 8-2(-2B+C)+1=5 \Rightarrow -2B+C=2$$

$$\Rightarrow C=0, \qquad 2B+C=-2 \Rightarrow B=-1$$

$$\int \frac{2x^2-x+2}{x^5+2x^3+x}dx = 2\int \frac{dx}{x} - \int \frac{2x}{x^2+1}dx - \int \frac{dx}{(x^2+1)^2}$$

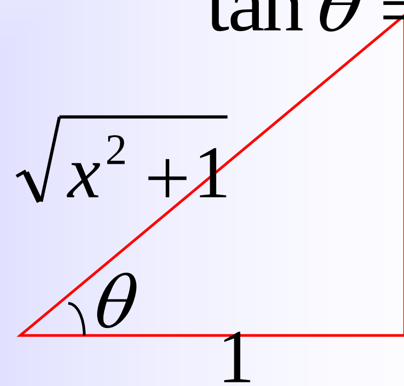
$$\int \frac{2x^2-x+2}{x^5+2x^3+x}dx = 2\ln|x| - \ln|x^2+1| - \int \frac{dx}{(x^2+1)^2} \qquad (1)$$

$$x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$$

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2\theta d\theta}{(\tan^2\theta+1)^2} = \int \frac{\sec^2\theta d\theta}{\sec^4\theta} = \int \cos^2\theta d\theta$$

$$=\int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta) = \frac{1}{2}(\theta + \sin\theta.\cos\theta)$$

$$\tan\theta = \frac{x}{1}$$



$$\sin\theta = \frac{x}{\sqrt{x^2+1}}, \quad \cos\theta = \frac{1}{\sqrt{x^2+1}}$$

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}(\tan^{-1}x + \frac{x}{\sqrt{x^2+1}}.\frac{1}{\sqrt{x^2+1}}) = \frac{1}{2}(\tan^{-1}x + \frac{x}{x^2+1}) \quad (2)$$

$$(1),(2) \Rightarrow I = 2\ln|x| - \ln|x^2+1| - \frac{1}{2}(\tan^{-1}x + \frac{x}{x^2+1}) + c$$

Note. **Case 5:** $\int \frac{f(x)}{g(x)}dx$ *where* degree $f(x) \geq$ degree $g(x)$,

divide $f(x)$ by $g(x)$;

$$f(x) = q(x).g(x) + r(x) \Rightarrow$$

$$\int \frac{f(x)}{g(x)}dx = \int \frac{q(x).g(x)+r(x)}{g(x)}dx = \int q(x)dx + \int \frac{r(x)}{g(x)}dx$$

Now use the method of Case 4, to find the second integral

Exercise. $$I=\int\frac{x^6-x^5+2x^4-2x^3+3x^2-2x+2}{x^5+2x^3+x}dx$$

Solution.

$$\begin{array}{r|l} & 2x^2-x+2 \\ \hline x^5+2x^3+x & x^6-x^5+2x^4-2x^3+3x^2-2x+2 \\ \cline{2-2} & x-1 \end{array}$$

$$I=\int\frac{(x^5+2x^3+x)(x-1)+2x^2-x+2}{x^5+2x^3+x}dx$$

$$=\int(x-1)dx+\int\frac{2x^2-x+2}{x^5+2x^3+x}dx=\frac{x^2}{2}-x+\int\frac{2x^2-x+2}{x^5+2x^3+x}dx$$

the latter integral was calculated in the previous example.