Lagrange Multipliers

To find the maximum or minimum of $f(x,y,z)$ under the condition $g(x,y,z)=0$, we solve the system of equations:

$$\begin{cases} \nabla f(x,,y,z) = \lambda.\nabla g(x,y,z) \\ g(x,y,z) = 0 \end{cases}$$

To find the maximum or minimum of $f(x,y,z)$ under the conditions $g(x,y,z)=0$ and $h(x,y,z)=0$, we solve the system of equations:

$$\begin{cases} \nabla f(x,,y,z) = \lambda.\nabla g(x,y,z) + \mu.\nabla h(x,y,z) \\ g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{cases}$$

Exercise. Find the closesest points to the origion on the surface $y^2 - x^2 = 1$.

Solution. We should find the minimum of $\sqrt{x^2 + y^2 + z^2}$, when $y^2 - x^2 = 1$. So it is enough to find the mininum of

$f(x, y, z) = x^2 + y^2 + z^2$ when $g(x, y, z) = y^2 - x^2 - 1 = 0$.

$$g(x, y, z) = y^2 - x^2 - 1 \quad \Rightarrow \nabla g(p) = (-2x)i + (2y)j$$

$$f(x, y, z) = x^2 + y^2 + z^2 \Rightarrow \nabla f(p) = (2x)i + (2y)j + (2z)k$$

$$\left.\begin{cases}\nabla f(p) = \lambda \nabla g(p) \\ g(p) = 0\end{cases}\right\} \Longrightarrow$$

$$\left.\begin{cases}(2x)i + (2y)j + (2z)k = \lambda\left((-2x)i + (2y)j \right) \\ y^2 - x^2 - 1 = 0\end{cases}\right\}$$

$$\Longrightarrow \begin{cases} 2x = -2\lambda x \Longrightarrow x = 0 \ \vee \ \lambda = -1 & (1) \\ 2y = 2\lambda y \Longrightarrow y = 0 \ \vee \ \lambda = 1 & (2) \\ z = 0 \\ y^2 - x^2 - 1 = 0 & (3) \end{cases}$$

$\lambda = -1 \xrightarrow{(2)} y = 0 \xrightarrow{(3)} x^2 = -1$ impossible. $(2) \Longrightarrow \lambda = 1$

$$\xrightarrow{(2)} \lambda = 1 \xrightarrow{(1)} x = 0 \xrightarrow{(3)} y^2 = 1 \Longrightarrow y = \pm 1$$

$\Longrightarrow$ the points are $(0,1,0)$ *or* $(0,-1,0)$

$$f\ (0,1,0) = f\ (0,-1,0) = 1.$$

To see that whether the points $(0,1,0)$ and $(0,-1,0)$ are minimum of maximum, we choose another point on the surface $y^2 - x^2 = 1$. The point $(\sqrt{3},2,1)$ is on this surface.

Now $f\ (\sqrt{3},2,1) = 8 > 1 = f\ (0,1,0) = f\ (0,-1,0)$.

So the points $(0,1,0)$ and $(0,-1,0)$ are the mimimum points.

There is no maximum for $x^2 + y^2 + z^2$ when $y^2 - x^2 - 1 = 0$, becuae there is no restrction for the value of z under the condition $y^2 - x^2 - 1 = 0$.

Exercise. Find the maximum of xyz , on the line of the intersection of the two planes

$$x - y - z = 8 \ , \ x + y + z = 4.$$

Solution. $f(x,y,z) = xyz, \ g(x,y,z) = x - y - z - 8, \ h(x,y,z) = x + y + z - 4.$

We should solve the system of equations:

$$\begin{cases} \nabla f(x,,y,z) = \lambda . \nabla g(x,y,z) + \mu . \nabla h(x,y,z) \\ g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{cases}$$

$$f(x,y,z) = xyz \Rightarrow \nabla f(x,y,z) = (yz)i + (xz)j + (xy)k .$$

$$g(x,y,z) = x + y + z - 4 = 0 \Rightarrow \nabla g(x,y,z) = i + j + k .$$

$$h(x,y,z) = x - y - z - 8 = 0 \Rightarrow \nabla h(x,y,z) = i - j - k .$$

$$\begin{cases} (yz)i + (xz)j + (xy)k = \lambda(i + j + k) + \mu(i - j - k) \\ x + y + z - 4 = 0 \\ x - y - z - 8 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} yz = \lambda + \mu & (1) \\ \left.\begin{aligned} xz &= \lambda - \mu & (2) \\ xy &= \lambda - \mu & (3) \end{aligned}\right\} \Rightarrow xz = xy \quad (6). \\ \left.\begin{aligned} x + y + z - 4 &= 0 & (4) \\ x - y - z - 8 &= 0 & (5) \end{aligned}\right\} \Rightarrow 2x = 12 \Rightarrow x = 6 \end{cases}$$

$$x = 6 \overset{(6)}{\Rightarrow} 6z = 6y \Rightarrow z = y. \ (7)$$

$$z=y,\ x=6 \overset{(4)}{\Rightarrow} 6+y+y-4=0 \Rightarrow y=-1 \overset{z=y}{\Rightarrow} z=-1.$$

The point is $(6,-1,-1)$. Note that the point (6,0,-2) is also

on the intersetion of the two planes.

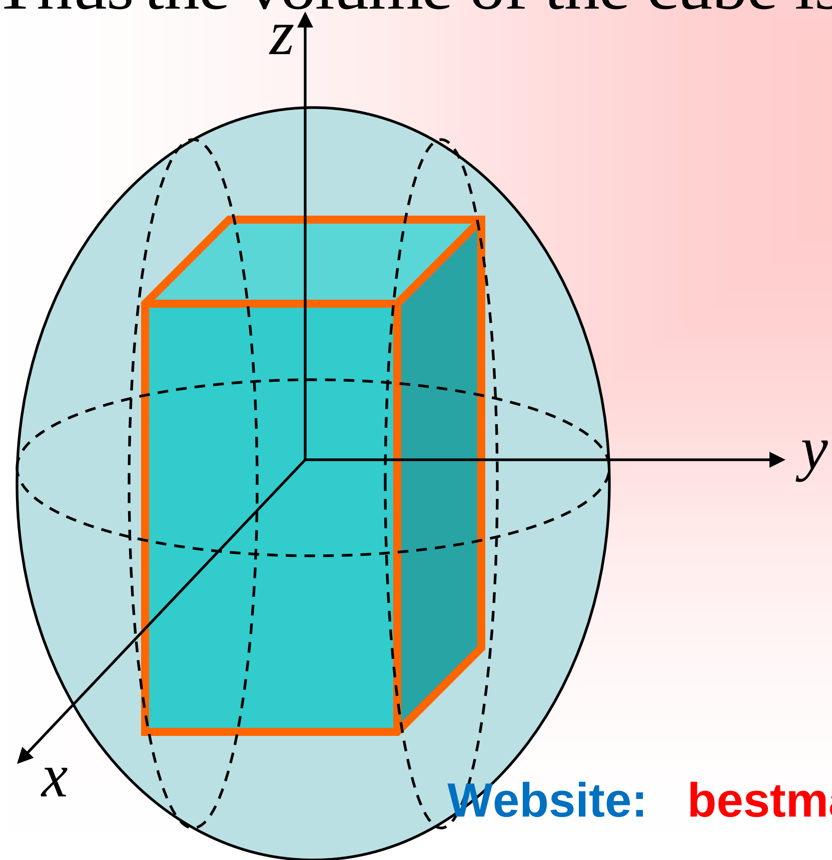
$f(6,-1,-1)=6\times(-1)\times(-1)=6>0=6\times0\times$(-2)=f(6,0,-2),

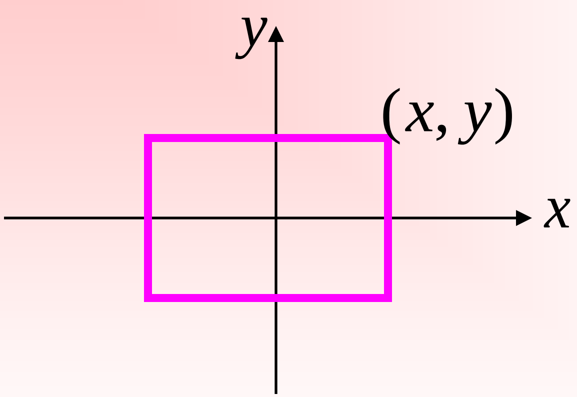
so $(6,-1,-1)$ is the maximum point.

Exercise. Find the dimensions of the rectangular parallelepiped of greatest volume that can be inscribed in the ellipsoid $x^2 + 9y^2 + z^2 = 9$. Assume that the edges are parallel to the coordinate axes.

Solution. Suppose $2x$ and $2y$ and $2z$ are the length, the width and the height of the cube. So the vertices of the cube are $(\pm x, \pm y, \pm z)$.

Thus the volume of the cube is $V = 2z \times 2y \times 2x = 8xyz$.





so it is enough to find the maximum of xyz ,

when $(\pm x)^2 + 9(\pm y)^2 + (\pm z)^2 = 9$

$$f(x,y,z) = xyz, \quad g(x,y,z) = x^2 + 9y^2 + z^2 - 9 = 0$$

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} yz = 2x\lambda & (1) \\ xz = 18y\lambda & (2) \\ xy = 2z\lambda & (3) \\ x^2 + 9y^2 + z^2 = 9 & (4) \end{cases}$$

$$(1),(2) \Rightarrow \frac{yz}{2x} = \frac{xz}{18y} \Rightarrow \frac{y}{x} = \frac{x}{9y} \Rightarrow 9y^2 = x^2 \quad (5)$$

$$(2),(3) \Rightarrow \frac{xz}{18y} = \frac{xy}{2z} \Rightarrow \frac{z}{9y} = \frac{y}{z} \Rightarrow 9y^2 = z^2 \quad (6)$$

$$(5),\ (6) \Rightarrow z^2 = x^2$$

$$x^2+x^2+x^2-9=0 \Rightarrow x=\pm\sqrt{3}$$

$$\Rightarrow z=\pm\sqrt{3}, \quad 9y^2=3 \Rightarrow y=\pm\frac{1}{\sqrt{3}} \Rightarrow 8\sqrt{3}\times\frac{1}{\sqrt{3}}\times\sqrt{3}=8\sqrt{3}$$

Maximum Volume