Integration By Parts

$$d(uv) = udv + vdu \Rightarrow uv = \int udv + \int vdu$$

$$\boxed{\int udv = uv - \int vdu}$$

Exercise. $\int Lnx\,dx$

$$\int u\,dv = uv - \int v\,du$$

Solution.

$u = Lnx \qquad dv = dx$

$du = \dfrac{dx}{x} \qquad v = x$

$$\Rightarrow \int Lnx\,dx = xLnx - \int x\frac{dx}{x}$$

$$= xLnx - \int dx = xLnx - x + C$$

Exercise. $\int x\,\tan^{-1}x\,dx$

$$I = \int x \tan^{-1} x \, dx$$

$$u = \tan^{-1} x \qquad dv = x\,dx$$

$$du = \frac{dx}{1+x^2} \qquad v = \frac{x^2}{2}$$

Solution.

$$I = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

$$\int \frac{x^2}{1+x^2}dx = \int \frac{(1+x^2)-1}{1+x^2}dx = \int dx - \int \frac{1}{1+x^2}dx$$

$$= x - \tan^{-1} x$$

$$\Rightarrow \boxed{I = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C}$$

Note. For calculating some integral Let I be the integral, and use By Part twice to find a relation in terms of I:

$$I = \int e^x \cos x dx$$

$$u = e^x \qquad dv = \cos x dx$$

$$du = e^x dx \qquad v = \sin x$$

Solution.

$$I = e^x \sin x - \int e^x \sin x dx$$

$$u = e^x \qquad dv = \sin x dx$$

$$du = e^x dx \qquad v = -\cos x$$

$$I = e^x \sin x + e^x \cos x - \int e^x \cos x dx = e^x(\sin x + \cos x) - I$$

$$2I = e^x(\sin x + \cos x) \Rightarrow \boxed{I = \frac{e^x(\sin x + \cos x)}{2} + C}$$

$$I=\int \sin(Lnx)dx$$

$u=\sin(\ln x)$ $\qquad dv=dx$

$du=\cos(Lnx)\frac{1}{x}dx$ $\qquad v=x$

Solution.

$$I=x\sin(\ln x)-\int x\cos(Lnx)\frac{1}{x}dx=x\sin(\ln x)-\int \cos(Lnx)dx$$

$u=\cos(Lnx)$ $\qquad dv=dx$

$du=-\sin(Lnx)\frac{1}{x}dx$ $\qquad v=x$

$$I=x\sin(\ln x)-(x\cos(Lnx)+\int x\sin(Lnx)\frac{1}{x}dx)$$

$$I=x(\sin(\ln x)-\cos(Lnx))-I\Rightarrow 2I=x(\sin(\ln x)-\cos(Lnx))$$

$$I=\frac{x(\sin(\ln x)-\cos(Lnx))}{2}+C$$

If $f(x)$ is a polynomials, for $\int f(x)g(x)dx$,

use the following table:

$f(x)$ and its derivatives		$g(x)$ and its integrals
$f(x)$		$g(x)$
	+	
$f_1(x)$		$g_1(x)$
	-	
$f_2(x)$		$g_2(x)$
	+	
$f_3(x)$		$g_3(x)$
	-	
$f_4(x)$		$g_4(x)$
$\vdots$		$\vdots$
0		$g_n(x)$

$$\int f(x)g(x)dx = f(x)g_1(x) - f_1(x)g_2(x) + f_2(x)g_3(x)$$
$$- \ldots + (-1)^{n-1} f_{n-1}(x)g_n(x) + C$$

Exercise. $\int x^4 \cos 2x dx$

Solution.

x^4 and its derivatives		$\cos 2x$ and its integrals
x^4	+	$\cos 2x$
$4x^3$	-	$\frac{1}{2}\sin 2x$
$12x^2$	+	$-\frac{1}{4}\cos 2x$
$24x$	-	$-\frac{1}{8}\sin 2x$
24	+	$\frac{1}{16}\cos 2x$
0		$\frac{1}{32}\sin 2x$

$$\int f(x)g(x)dx = \frac{1}{2}x^4 \sin 2x + x^3 \cos 2x +$$
$$12 \times \frac{-1}{8} x^2 \sin 2x - 24 \times \frac{1}{16} x \cos 2x + 24 \times \frac{1}{32} \sin 2x + C$$

$$1)\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$2)\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$3)\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$4)\int \csc ax\,dx = \frac{1}{a}\ln|\csc ax - \cot ax| + C$$

$$5)\int \tan^n ax \cdot \sec^2 ax\,dx = \frac{\tan^{n+1} ax}{a(n+1)} + C$$

$$6)\int \cot^n ax \cdot \csc^2 ax dx = -\frac{\cot^{n+1} ax}{a(n+1)} + C$$

$$Proof.\ \int \sec ax dx = \int \frac{\sec ax(\sec ax + \tan ax)}{\sec ax + \tan ax} dx$$

$$= \int \frac{\sec^2 ax + \sec ax \cdot \tan ax}{\sec ax + \tan ax} dx = \frac{1}{a} \ln|\sec ax + \tan ax| + C$$

$\int \sec^n ax dx$ و $\int \csc^n ax dx$

(a) n is odd. Use By Parts as follows:

$$I = \int \sec^n ax dx = \int \sec^{n-2} ax . \sec^2 ax dx$$

$$u = \sec^{n-2} ax \qquad dv = \sec^2 ax dx$$

Exercise. $\int \csc^3 2x dx$

Solution.

$$\int \csc^3 2x\,dx$$

$$I = \int \csc^3 2x\,dx = \int \csc 2x.\csc^2 2x\,dx$$

$$u = \csc 2x \Rightarrow \qquad du = -2\csc 2x.\cot 2x\,dx,$$

$$dv = \csc^2 2x\,dx \Rightarrow \qquad v = -\frac{\cot 2x}{2}$$

$$I = -\csc 2x.\frac{\cot 2x}{2} - \int \cot^2 2x.\csc 2x\,dx$$

$$I = -\csc 2x.\frac{\cot 2x}{2} - \int (\csc^2 2x - 1)\csc 2x\,dx$$

$$I = -\csc 2x.\frac{\cot 2x}{2} - I + \int \csc 2x\,dx$$

$$I = -\csc 2x.\frac{\cot 2x}{2} - I + \frac{1}{2} Ln|\csc 2x - \cot 2x|$$

$$\boxed{I = \frac{1}{2}\left(-\csc 2x.\frac{\cot 2x}{2} + \frac{1}{2} Ln|\csc 2x - \cot 2x|\right) + c}$$

$$\int \sec^n axdx \quad و \quad \int \csc^n axdx$$

n is even. Let $n-2=2k$.

$$\int \sec^n axdx = \int \sec^{n-2} ax \cdot \sec^2 axdx =$$

$$\int \sec^{2k} ax \cdot \sec^2 axdx = \int (\sec^2 ax)^k \cdot \sec^2 axdx$$

$$= \int (1+\tan^2 ax)^k \cdot \sec^2 axdx$$

Now exapnd $(1+\tan^2 ax\,)^k$ and use:

$$\int \tan^n ax \cdot \sec^2 axdx = \frac{\tan^{n+1} ax}{a(n+1)} + C$$

$$\int \cot^n ax \cdot \csc^2 axdx = -\frac{\cot^{n+1} ax}{a(n+1)} + C$$

Exercise.

The region bounded by $y = \csc^3 2x$, $x = \dfrac{\pi}{4}$ *and* $x = \dfrac{\pi}{12}$ *and* $x-$axis is rotated about the x-axis.

Find the volume of revolution.

$$I=\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\csc^6 2x\,dx$$

Solution.

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\csc^6 2x\,dx=\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\csc^4 2x.\csc^2 2x\,dx=$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}(1+\cot^2 2x)^2\csc^2 2x\,dx=$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}(\csc^2 2x+\cot^4 2x.\csc^2 2x+2\cot^2 2x.\csc^2 2x)\,dx$$

$$=(-\frac{1}{2}\cot 2x-\frac{1}{10}\cot^5 2x-\frac{1}{3}\cot^3 2x)\,|_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$