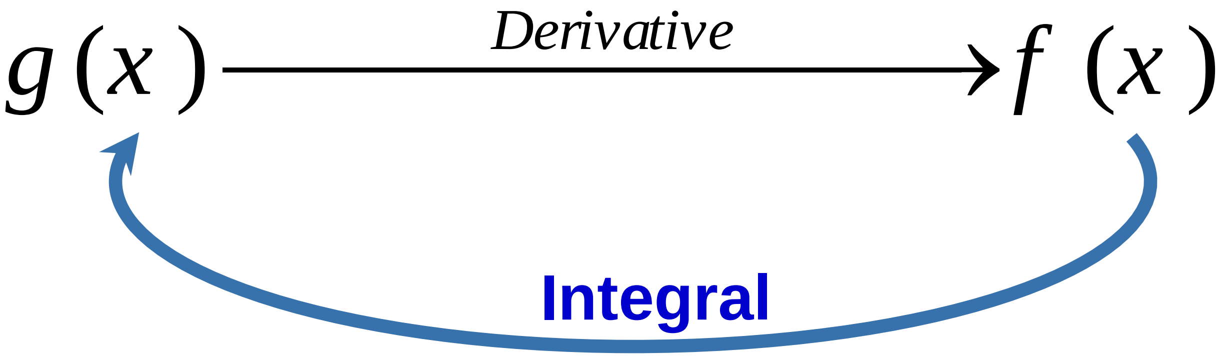
Anti-Derivative or Indefinite Integral

If $g'(x) = f(x)$, **then we write** $\int f(x)\,dx = g(x) + c$



Formulas:

$$1)\int \left(f(x) \pm g(x)\right)dx = \int f(x)dx \pm \int g(x)dx$$

$$2)\int cf(x)dx = c\int f(x)dx$$

$$3)\int x^r dx = \frac{x^{r+1}}{r+1} + C, \quad -1 \neq r \in Q$$

4) (Change of variable) If $\int f(x)dx = g(x)$ **, then**

$\int f(u)du = g(u) + c,$ where u is a fuction of x.

Exercise.

1) $\int \frac{x^3 - 3x + 11}{2\sqrt[3]{x}} dx$

2) $\int \left(x^2 - 6x + 9\right)^{\frac{8}{3}} dx$

3) $\int \frac{x^3 .dx}{\sqrt{1 - 2x^2}}$

4) $\int \frac{dx}{\sqrt{x\sqrt{x} + x}}$

Solution.

$$1)I=\int\frac{x^3-3x+11}{2\sqrt[3]{x}}dx$$

$$1)I=\frac{1}{2}\int x^{\frac{8}{3}}dx-\frac{3}{2}\int x^{\frac{2}{3}}dx+\frac{11}{2}\int x^{\frac{-1}{3}}dx=\frac{3}{22}x^{\frac{11}{3}}-\frac{9}{10}x^{\frac{5}{3}}+\frac{33}{4}x^{\frac{2}{3}}+c$$

$$2)I=\int\left(x^2-6x+9\right)^{\frac{8}{3}}dx$$

$$2)I=\int\left(\left(x-3\right)^2\right)^{\frac{8}{3}}dx=\int\left(x-3\right)^{\frac{16}{3}}dx=\frac{\left(x-3\right)^{\frac{19}{3}}}{\frac{19}{3}}+c=\frac{3\left(x-3\right)^{\frac{19}{3}}}{19}+c$$

$$3)\,I=\int\frac{x^3.dx}{\sqrt{1-2x^2}}=\int x^2\frac{xdx}{\sqrt{1-2x^2}}\qquad u=\sqrt{1-2x^2}$$

$$\Rightarrow du=\frac{-4xdx}{2\sqrt{1-2x^2}}\Rightarrow -\frac{1}{2}du=\frac{xdx}{\sqrt{1-2x^2}},\qquad u^2=1-2x^2\Rightarrow x^2=\frac{1-u^2}{2}$$

$$I=\int\frac{x\cdot x^2.dx}{\sqrt{1-2x^2}}=\int\frac{1-u^2}{2}\times\left(\frac{-1}{2}\right)du=\frac{-1}{4}\int\left(1-u^2\right)du=\frac{-1}{4}u+\frac{u^3}{12}+c$$

$$=\frac{-1}{4}\sqrt{1-2x^2}+\frac{\sqrt{1-2x^2}^{\,3}}{12}+c$$

$$4)\ I=\int\frac{dx}{\sqrt{x\sqrt{x}+x}}=\int\frac{dx}{\sqrt{x}\sqrt{\sqrt{x}+1}}\qquad u=\sqrt{x}+1\Rightarrow du=\frac{dx}{2\sqrt{x}}$$

$$4)\ I=2\int\frac{dx}{2\sqrt{x}\sqrt{\sqrt{x}+1}}=2\int\frac{du}{\sqrt{u}}=2\int u^{-\frac{1}{2}}du=4u^{\frac{1}{2}}+C=4\sqrt{\sqrt{x}+1}+C$$

Integral of Trigonometric Functions

Formulas:

1) $$\int \sin ax\,dx = \frac{-\cos ax}{a} + C$$

2) $$\int \cos ax\,dx = \frac{\sin ax}{a} + C$$

3) $$\int \sin^n ax \, . \cos ax\,dx = \frac{\sin^{n+1} ax}{a(n+1)} + C$$

4) $$\int \cos^n ax \, . \sin ax\,dx = \frac{-\cos^{n+1} ax}{a(n+1)} + C$$

5) $$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C \qquad (\tan x)' = \sec^2 x$$

Formulas:

$$(\cot x)' = -\csc^2 x$$

$$7)\int \sec^n ax \,.\tan ax\,dx = \int \sec^{n-1} ax\,.\sec ax\,.\tan ax\,dx = \frac{\sec^n ax}{a.n} + C$$

$$8)\int \csc^n ax\,.\cot ax\,dx = \frac{-\csc^n ax}{a.n} + C$$

$$(\sec x)' = \sec x\,.\tan x$$

$$9)\int \tan^n ax\,.\sec^2 ax\,dx = \frac{\tan^{n+1} ax}{a(n+1)} + C$$

$$(\csc x)' = -\csc x\,.\cot x$$

$$10)\int \cot^n ax\,.\csc^2 ax\,dx = \frac{-\cot^{n+1} ax}{a(n+1)} + C$$

Exercise.

1) $$\int \frac{\sin 3\sqrt{t}}{\sqrt{t}}\,dt$$

2) $$\int \frac{\cos\frac{x}{4}}{\sqrt{\sin\frac{x}{4}}}\,dx$$

3) $$\int \sin^8(\sin x).\cos(\sin x).\cos x\,dx$$

4) $$\int \frac{\sec^5\sqrt{x}.\tan\sqrt{x}}{\sqrt{x}}\,dx$$

5) $$\int \tan^7(\sec x).\sec^2(\sec x).\sec x.\tan x\,dx$$

6) $$\int \frac{\sec^5(\csc x).\tan(\csc x).\cos x}{1-\cos 2x}\,dx$$

Solution.

$$1)\ 2\int \frac{\sin 3\sqrt{t}}{2\sqrt{t}}dt \qquad u=\sqrt{t} \Leftrightarrow du=\frac{1}{2\sqrt{t}}dt$$

$$=2\int \sin 3u\,du=\frac{-2}{3}\cos 3\sqrt{t}+C$$

$$2)\int \frac{\cos\frac{x}{4}}{\sqrt{\sin\frac{x}{4}}}dx \qquad u=\sin\frac{x}{4} \Rightarrow du=\frac{1}{4}\cos\frac{x}{4}dx$$

$$2)=4\int \frac{\frac{1}{4}\cos\frac{x}{4}}{\sqrt{\sin\frac{x}{4}}}dx=4\int\frac{1}{\sqrt{u}}du=4\int u^{\frac{-1}{2}}=4\frac{u^{\frac{1}{2}}}{\frac{1}{2}}=8\sqrt{\sin\frac{x}{4}}+c$$

$$4)\int \sin^{8}(\sin x).\cos(\sin x).\cos x\,dx \qquad \sin x = u \Rightarrow \cos x\,dx = du$$

$$= \int \sin^{8} u.\cos u.du = \frac{\sin^{9} u}{9} + c = \frac{\sin^{9}(\sin x)}{9} + c$$

$$5)\int \tan^{7}(\sec x).\sec^{2}(\sec x).\sec x.\tan x\,dx$$

$$\sec x = u \Rightarrow \sec x.\tan x\,dx = du$$

$$= \int \tan^{7} u.\sec^{2} u\,du = \frac{\tan^{8} u}{8} + c$$

$$6)\int \frac{\sec^5(\csc x).\tan(\csc x).\cos x}{1-\cos 2x}dx$$

$$= -\int \frac{\sec^5(\csc x).\tan(\csc x).\cos x}{-2\sin^2 x}dx$$

$$u = \csc x \Rightarrow du = -\csc x.\cot x dx = \frac{-\cos x}{\sin^2 x}dx$$

$$= \frac{-1}{2}\int \sec^5 u.\tan u.du$$

$$= \frac{-1}{2}\frac{\sec^5 u}{5} = \frac{-\sec^5 u}{10} + c \quad = \frac{-\sec^5(\csc x)}{10} + c$$

$$\int \cos^n ax\,dx \quad \& \quad \int \sin^n ax\,dx$$

(a) n is an odd natural number

$$\int \sin^n ax\,dx = \int \sin^{n-1} ax.\sin ax\,dx = \int \sin^{2k} ax.\sin ax\,dx \qquad n-1=2k$$

$$= \int \left(\sin^2 ax\right)^k .\sin ax\,dx = \int \left(1-\cos^2 ax\right)^k .\sin ax\,dx = \ldots$$

Now expand $\left(1-\cos^2 ax\right)^k$ **and use Formulas (1) and (4) to find the integrals**

Exercise. $\int x\cos^5 x^2\,dx$

Solution.

$$\int x\cos^5 x^2\,dx$$

$$=\frac{1}{2}\int \cos^5 u\,du \qquad u=x^2 \Leftrightarrow du=2x\,dx \Rightarrow \frac{1}{2}du=x\,dx$$

$$=\frac{1}{2}\int \cos^4 u.\cos u\,du=\frac{1}{2}\int\left(\cos^2 u\right)^2.\cos u\,du=\frac{1}{2}\int\left(1-\sin^2 u\right)^2.\cos u\,du$$

$$=\frac{1}{2}\int \cos u\,du+\frac{1}{2}\int \sin^4 u.\cos u\,du-\int \sin^2 u.\cos u\,du$$

$$=\frac{1}{2}\sin x^2+\frac{1}{10}\sin^5 x^2-\frac{1}{3}\sin^3 x^2+C$$

(b) n is an even natural number $(n = 2k)$

$$\int \cos^n ax = \int \cos^{2k} ax dx = \int \left(\cos^2 ax\right)^k dx = \int \left(\frac{1+\cos 2ax}{2}\right)^k dx$$

Now expand $\left(\frac{1+\cos 2ax}{2}\right)^k$ **and use method (a) or repeat this method again to calculate the integrals**

Note: Expansion Formulas

$$\sin^2 ax = \frac{1-\cos 2ax}{2}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$(1+a)^n = 1 + na + \binom{n}{2}a^2 + \binom{n}{3}a^3 + \binom{n}{4}a^4 + \ldots + na^{n-1} + a^n$$

$$(1-a)^n = 1 - na + \binom{n}{2}a^2 - \binom{n}{3}a^3 + \binom{n}{4}a^4 - \ldots + (-1)^{n-1}na^{n-1} + (-1)^n a^n$$

Exercise. $\int \sin^6 3x\,dx$

Website: bestmathtutor.ca **Contact Number: 778-882-4636**

Solution. $$I=\int \sin^6 3x\,dx$$

$$=\int\left(\sin^2 3x\right)^3 dx=\int\left(\frac{1-\cos 6x}{2}\right)^3 dx=\frac{1}{8}\int\left(1-\cos 6x\right)^3 dx$$

$$=\frac{1}{8}\int(1-3\cos 6x+3\cos^2 6x-\cos^3 6x)dx$$

$$=\frac{1}{8}\int 1dx-\frac{3}{8}\int \cos 6x dx+\frac{3}{8}\int \cos^2 6x dx-\frac{1}{8}\int \cos^3 6x dx$$

$$=\frac{x}{8}-\frac{1}{16}\sin 6x+\frac{3}{8}\int \cos^2 6x dx-\frac{1}{8}\int \cos^3 6x dx$$

$$\int \cos^2 6x dx = \int \frac{1+\cos 12x}{2} dx = \frac{1}{2}\int (1+\cos 12x) dx$$

$$= \frac{x}{2} + \frac{\sin 12x}{24}$$

$$\int \cos^3 6x dx = \int \cos^2 6x.\cos 6x dx = \int \left(1-\sin^2 6x\right)\cos 6x dx$$

$$\int (\cos 6x - \sin^2 6x.\cos 6x) dx = \frac{\sin 6x}{6} - \frac{\sin^3 6x}{3\times 6}$$

$$\Rightarrow I = \frac{x}{8} - \frac{1}{16}\sin 6x + \frac{3}{8}\left(\frac{x}{2} + \frac{\sin 12x}{24}\right) - \frac{1}{8}\left(\frac{\sin 6x}{6} - \frac{\sin^3 6x}{18}\right) + C$$

$$\int \tan^n ax\,dx, \int \cot^n ax\,dx$$

$$\int \tan^n ax\,dx = \int \tan^{n-2} ax.\tan^2 ax\,dx = \int \tan^{n-2} ax.\left(\sec^2 ax - 1\right)dx$$

$$= \int \tan^{n-2} ax.\sec^2 ax\,dx - \int \tan^{n-2} ax\,dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int \tan^{n-2} ax\,dx$$

Now continue this process for $\int \tan^{n-2} ax\,dx$

Exercise. $\int \cot^6 \sqrt{3}x\,dx$

Solution.

$$I = \int \cot^6 \sqrt{3}x\,dx$$

$$I = \int \cot^4 \sqrt{3}x \cdot \cot^2 \sqrt{3}x\,dx =$$

$$\int \cot^4 \sqrt{3}x \cdot (\csc^2 \sqrt{3}x - 1)\,dx = \int \cot^4 \sqrt{3}x \cdot \csc^2 \sqrt{3}x\,dx$$

$$-\int \cot^4 \sqrt{3}x\,dx = -\frac{\cot^5 \sqrt{3}x}{5\sqrt{3}} - \int \cot^4 \sqrt{3}x\,dx$$

$$\int \cot^4 \sqrt{3}x dx = \int \cot^2 \sqrt{3}x . \cot^2 \sqrt{3}x dx = \int \cot^2 \sqrt{3}x . (\csc^2 \sqrt{3}x - 1) dx$$

$$= \int \cot^2 \sqrt{3}x \cdot \csc^2 \sqrt{3}x \, dx - \int \cot^2 \sqrt{3}x \, dx = -\frac{\cot^3 \sqrt{3}x}{3\sqrt{3}} - \int \cot^2 \sqrt{3}x \, dx$$

$$\int \cot^2 \sqrt{3}x = \int (\csc^2 \sqrt{3}x - 1) dx = -\frac{\cot \sqrt{3}x}{\sqrt{3}} - x + C$$

$$\Rightarrow I = -\frac{\cot^5 \sqrt{3}x}{5\sqrt{3}} + \frac{\cot^3 \sqrt{3}x}{3\sqrt{3}} - \frac{\cot \sqrt{3}x}{\sqrt{3}} + x + C$$