**Definition.** **Every point in $R^n$ is of the form**

$$(x_1, x_2, x_3, \ldots, x_n), \quad x_i \in \mathbb{R}$$

**For every n-variable function f,** $f: R^n \to R$

**that is** $w \in R^n$ $\quad f(w) \in R$

**and** $w_1 = w_2 \Rightarrow f(w_1) = f(w_2)$

$$D_f = \{w \in R^n \mid f(w) \text{ is defined}\}$$

$$R_f = \{t \in R \mid \text{there exists } w \in R^n \text{ with } f(w) = t\}$$

Evidently $D_f \subseteq R^n$, $\quad R_f \subseteq R$

# Exercise.

## Find the domain and the rage of the following function and graph the domain and the range.

$$a)\ f(x,y)=\sqrt{9-(x^2+y^2)}$$

$$b)\ f(x,y)=\frac{\sqrt{25-(x^2+y^2)}}{x}$$

$$c)\ f(x,y)=\sqrt{\frac{x-y}{x+y}}$$

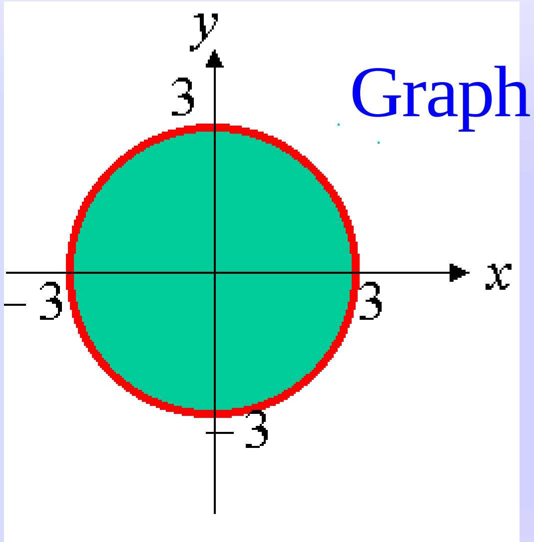
$$d)\ f(x,y,z)=\sqrt{z-(x^2+y^2+1)}$$

$$e)\ f(x,y,z)=\sqrt{(x^2+y^2)-9}$$

**Solution.**

$$a)\ f(x,y)=\sqrt{9-(x^2+y^2)}$$

$$D_f=\{(x,y)\in R^2 \mid 9-(x^2+y^2)\geq 0\}=\{(x,y)\in R^2 \mid x^2+y^2\leq 9\}$$



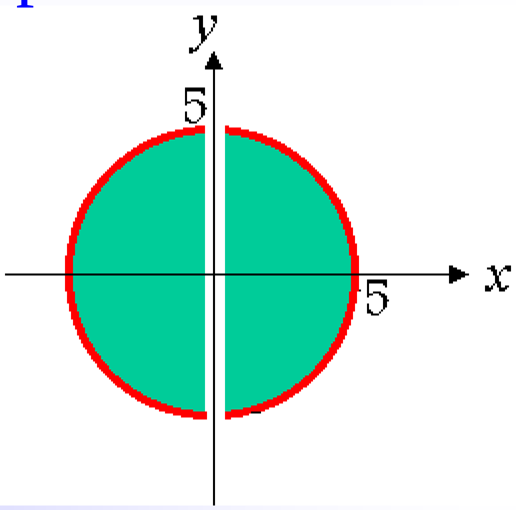

Graph of domain

Graph of range

$$R_f=[0,3]$$

$$b)\ f(x,y)=\frac{\sqrt{25-(x^2+y^2)}}{x}$$

Graph of domain




$$D_f=\{(x,y)\in R^2 \mid 25-(x^2+y^2)\geq 0,\ x\neq 0\}$$

$$=\{(x,y)\in R^2 \mid x^2+y^2\leq 25,\ x\neq 0\}$$

Put $y=0$

$$y=0\Rightarrow w=\frac{\sqrt{25-x^2}}{x}\Rightarrow x^2+w^2x^2-25=0\Rightarrow x=\pm\sqrt{\frac{25}{1+w^2}}$$

$$\Rightarrow R_f=R$$

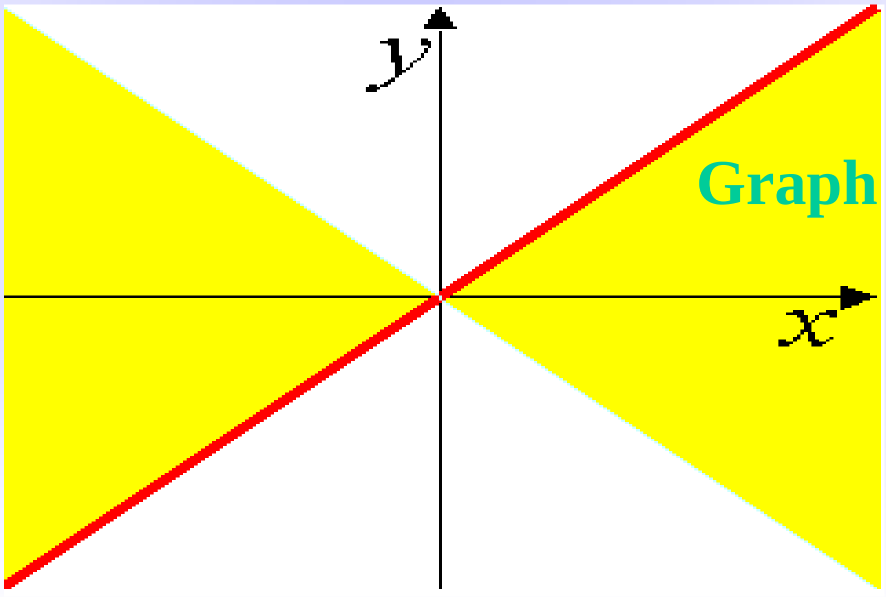
Graph of range

$$c)\ f(x,y)=\sqrt{\frac{x-y}{x+y}}$$

$$D_f=\{(x,y)\in R^2 \mid \frac{x-y}{x+y}\geq 0,\ x+y\neq 0\}$$

$$=\{(x,y)\in R^2 \mid (x-y\geq 0,\ x+y>0)\ \textbf{or}\ \ (x-y<0,\ x+y<0)\}$$

$$=\{(x,y)\in R^2 \mid (y\leq x,\ y>-x)\ \textbf{or}\ \ (y>x,\ y<-x)\}$$




**Graph of domain**

$$c)\ f(x,y)=\sqrt{\frac{x-y}{x+y}}$$

$$\sqrt{\frac{x-y}{x+y}}\geq 0\Rightarrow R_f\subseteq[0,+\infty)\qquad(1)$$

$$y=1\Rightarrow w=\sqrt{\frac{x-1}{x+1}}\Rightarrow w^2=\frac{x-1}{x+1}=\frac{x+1-2}{x+1}=1-\frac{2}{x+1}\Rightarrow$$

$$\frac{2}{x+1}=1-w^2\Rightarrow x+1=\frac{2}{1-w^2}\Rightarrow x=\frac{2}{1-w^2}-1,\ w\geq 0$$

$$\Rightarrow w\in[0,+\infty)-\{1\}\Rightarrow[0,+\infty)-\{1\}\subseteq R_f\qquad(2)$$

$$f(1,0)=1\Rightarrow 1\in R_f\qquad(3)$$

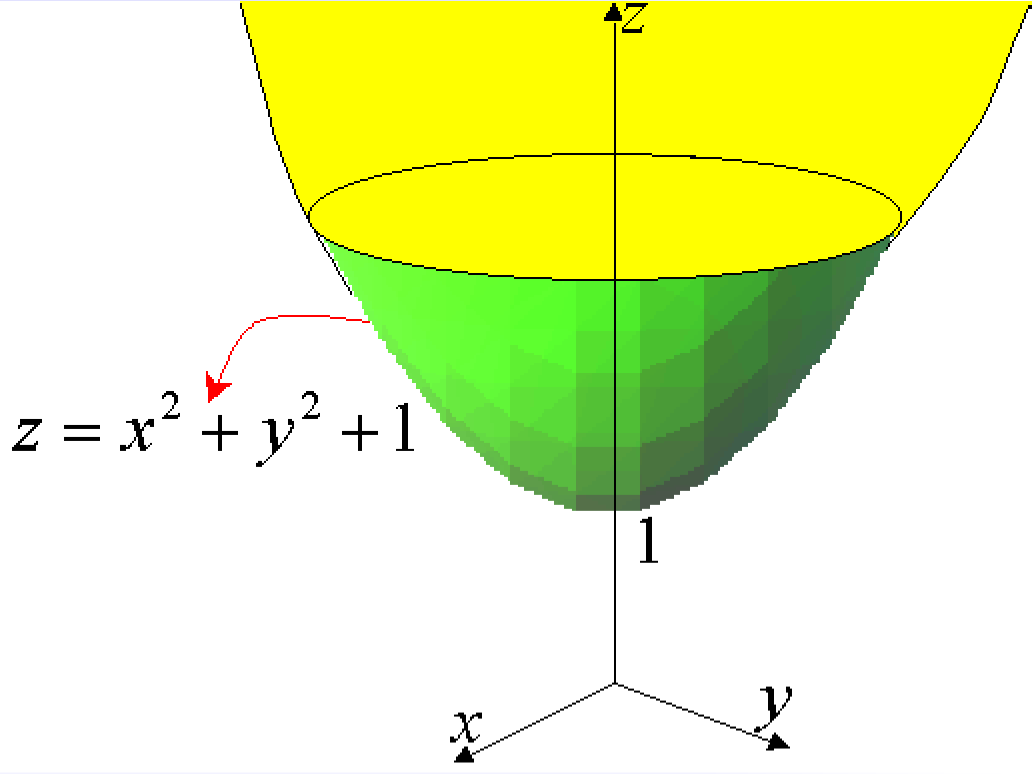
**Graph of range**


$$(1),(2),(3)\Rightarrow R_f=[0,+\infty)$$

$$d)\ f(x,y,z)=\sqrt{z-(x^2+y^2+1)}$$

$$D_f=\{(x,y,z)\in R^3 \mid z-(x^2+y^2+1)\geq 0\}$$

$$D_f=\{(x,y,z)\in R^3 \mid z\geq x^2+y^2+1\}$$



$$d)\ f(x,y,z)=\sqrt{z-(x^2+y^2+1)}$$

$$\sqrt{z-(x^2+y^2+1)}\geq 0 \Rightarrow R_f \subseteq [0,+\infty) \qquad (1)$$

$$x=0,\ y=0 \Rightarrow w=\sqrt{z-1} \Rightarrow w^2=z-1$$

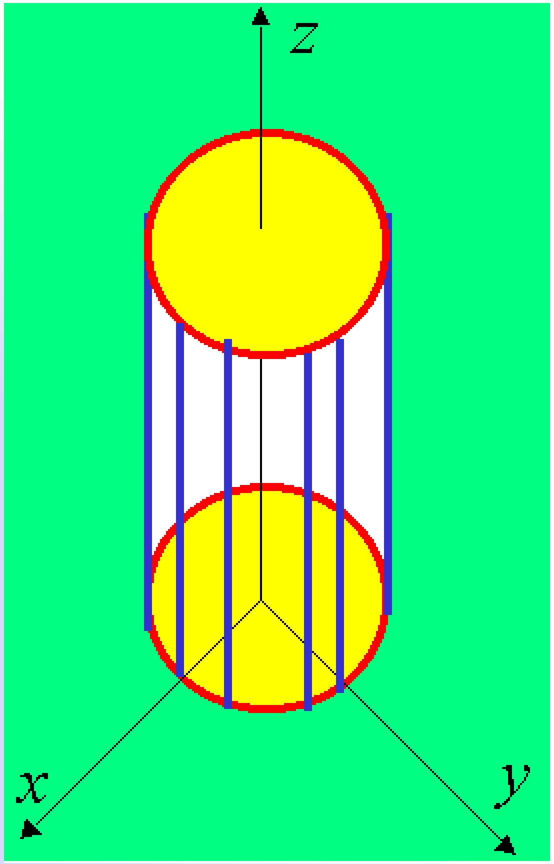
$$\Rightarrow z=w^2+1, \quad w\geq 0 \Rightarrow w\in[0,+\infty) \Rightarrow [0,+\infty)\subseteq R_f \qquad (2)$$

$$(1),(2) \Rightarrow R_f=[0,+\infty)$$

$$e)\ f(x,y,z)=\sqrt{(x^2+y^2)-9}$$

$$D_f=\{(x,y,z)\in R^3 \mid (x^2+y^2)-9\geq 0\}$$

$$D_f=\{(x,y,z)\in R^3 \mid x^2+y^2\geq 9\}$$




$$R_f=[0,3]$$

**Definition.** **If *f* is a two-variable function, then the graph of *f* is the graph of the surface** $z = f(x, y)$.

**Exercise.**

Graph the function $f(x, y) = -\sqrt{4y^2 - 16y - 9x^2 + 18x + 32} - 3$

**Solution.**

$$z+3=-\sqrt{4y^2-16y-9x^2+18x+32}$$

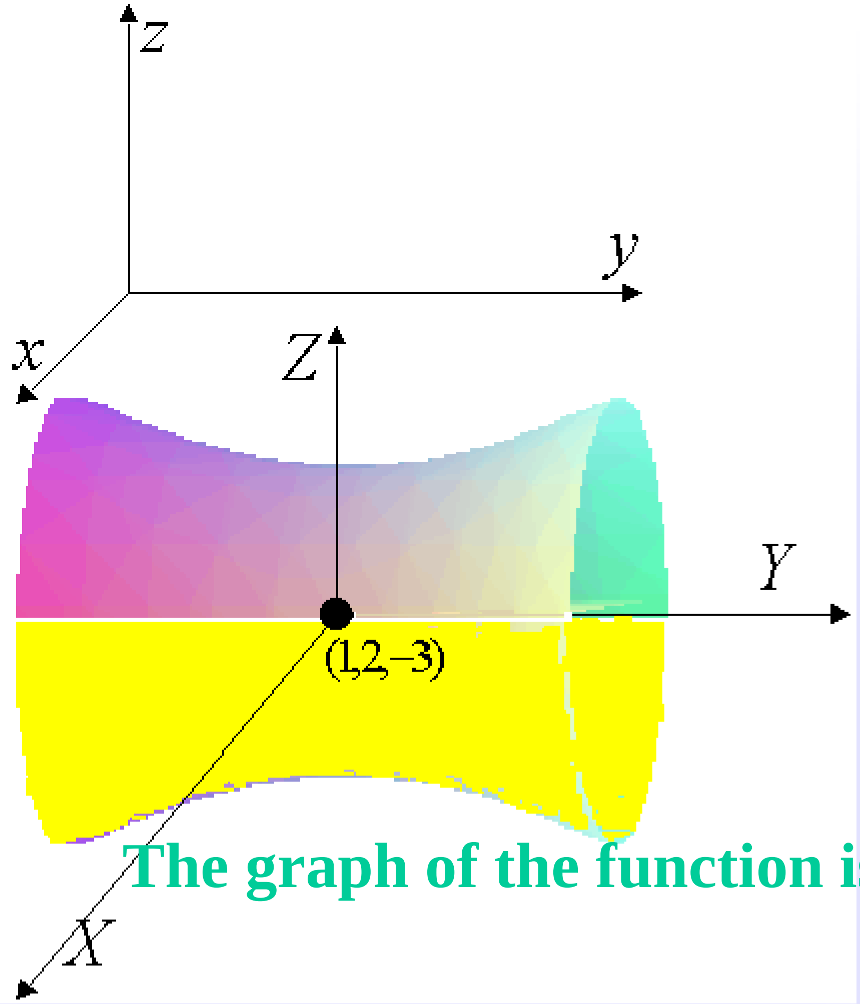
$$(z+3)^2=4(y^2-4y)-9(x^2-2x)+32,\ z+3\le 0$$

$$9(x-1)^2+(z+3)^2-4(y-2)^2=25,\ z+3\le 0$$

**Now translate the origin to the point** $(1,2,-3)$ **to get:**

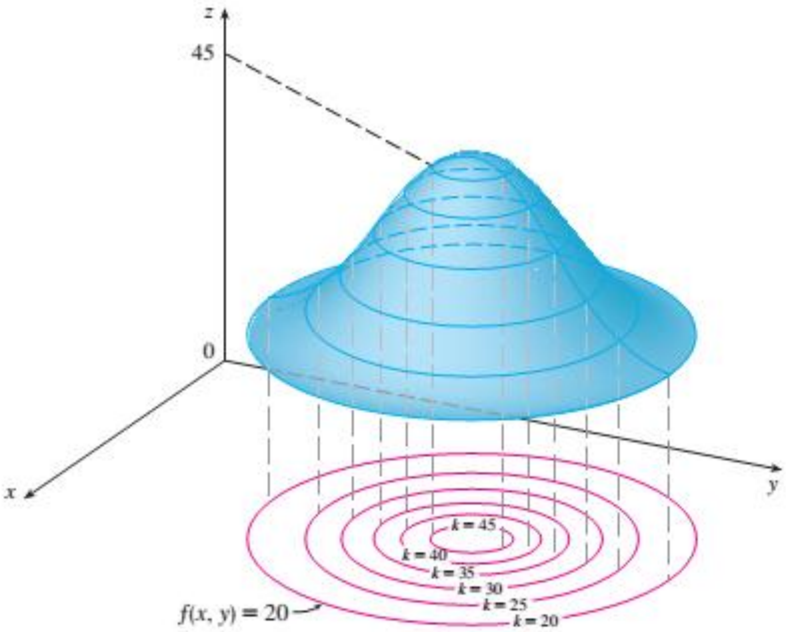
$$\Rightarrow 9X^2+Z^2-4Y^2=25,\ Z\le 0$$

**Elliptic Hyperbola of one sheet**




**The graph of the function is the yellow part**

**Definition.** **For the function z=f(x,y), the curve f(x,y)=k, where k is constant is called a level curve.**



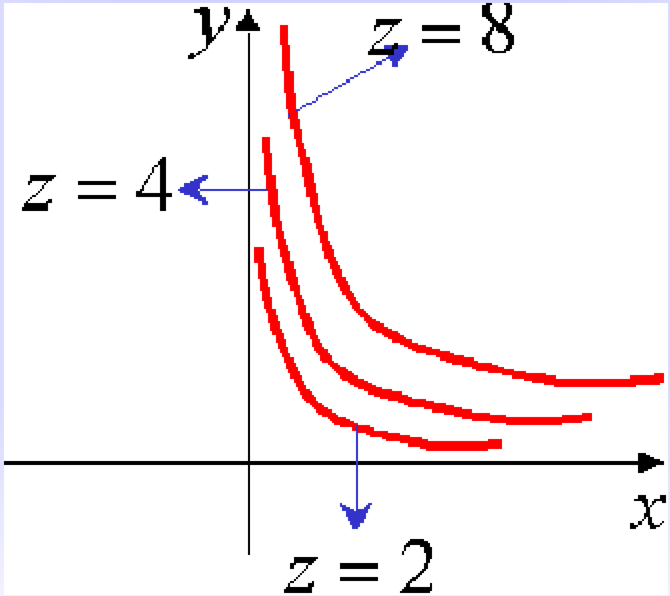

## Exercise.

**Sketch several level curves of the function.**

$$f(x,y) = 2x^{\frac{1}{2}}y^{\frac{1}{2}}$$

**Solution.**

$$2x^{\frac{1}{2}}y^{\frac{1}{2}} = k \Rightarrow \begin{cases} 4xy = k^2 \\ x, y > 0 \end{cases} \Rightarrow y = \frac{k^2}{4x}, \quad x > 0$$



**Exercise.**

**Find a level curve of the function** $f(x,y)=\sum_{n=0}^{+\infty}\left(\frac{x}{y}\right)^{n}$ **passing the point** $(1,2)$

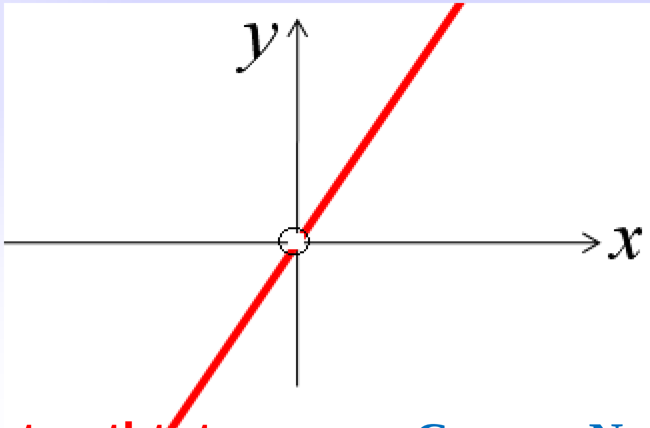
**Solution.**

$$f(x,y)=k,\ f(1,2)=k\Rightarrow f(x,y)=f(1,2)\Rightarrow\sum_{n=0}^{+\infty}\left(\frac{x}{y}\right)^n=\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n=\frac{1}{1-\frac{1}{2}}=2$$

$$\Rightarrow\sum_{n=0}^{+\infty}\left(\frac{x}{y}\right)^n=2$$

If $\left|\frac{x}{y}\right|<1$, then $\sum_{n=0}^{+\infty}\left(\frac{x}{y}\right)^n=\frac{1}{1-\frac{x}{y}}$ (Geometric series)

$$\Rightarrow\frac{1}{1-\frac{x}{y}}=2\Rightarrow 1-\frac{x}{y}=\frac{1}{2}\Rightarrow\frac{x}{y}=\frac{1}{2}\Rightarrow y=2x,\ \ y\neq 0$$




$y=2x,\ \ y\neq 0$

**Definition.** **For the function w=f(x,y,z), the surface f(x,y,z)=k, where k is constant is called a level surface.**

**Exercise.**

**Sketch the level surfaces of the function** $f(x, y, z) = z - x^2 - y^2$

**at 2 and 5**

**Solution.** $k=2 \Rightarrow z-x^2-y^2=2 \Rightarrow z-2=x^2+y^2$

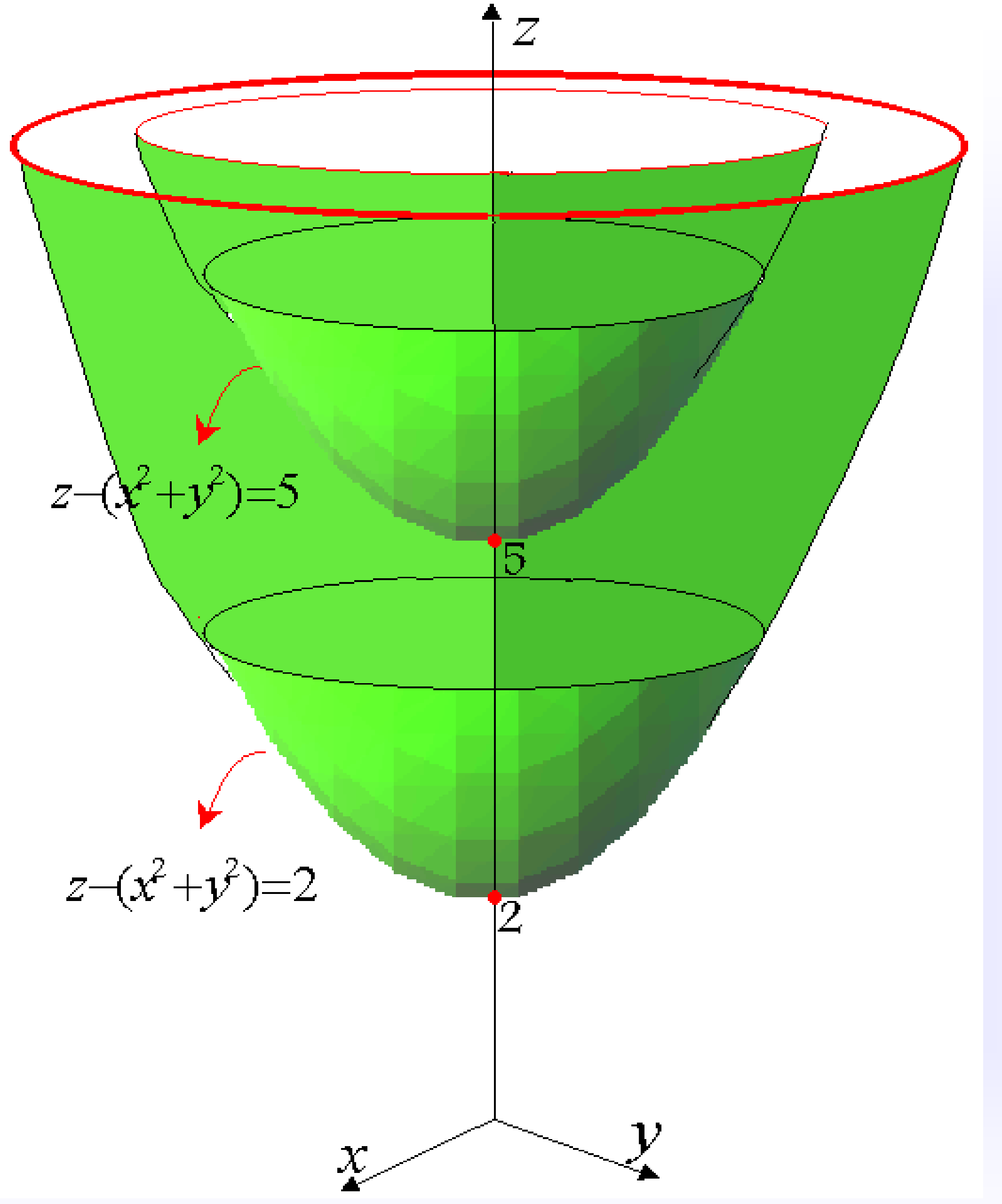
**Translate the origin to the point** $(0,0,2)$ **to get:** $Z=X^2+Y^2$

**This surface is given by revolving the cure** $Y=\sqrt{Z}$ **about z-axis**

**Similarly** $k=5 \Rightarrow z-x^2-y^2=5 \Rightarrow z-5=x^2+y^2$

**Translate the origin to the point** $(0,0,5)$ **to get:** $Z=X^2+Y^2$

**This surface is given by revolving the cure** $Y=\sqrt{Z}$ **about z-axis**


z
$z-(x^2+y^2)=5$
5
$z-(x^2+y^2)=2$
2
x
y