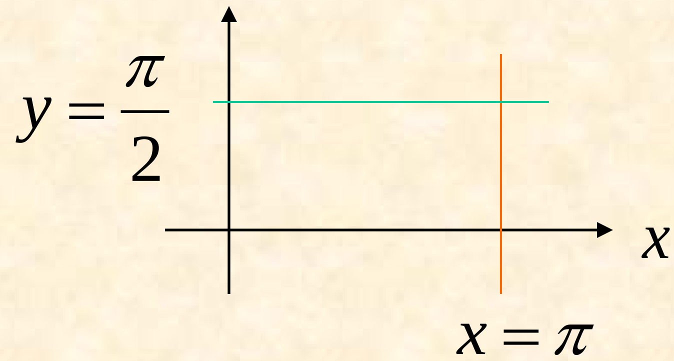
Double Integral

Exercises.

Find $\iint_D (\sin x + \sin y)\,dA$ **, where D is a rectangle with the vertices**

y

$$(0,\frac{\pi}{2}), (\pi,0), (\pi,\frac{\pi}{2}), (0,0)$$



Solution.

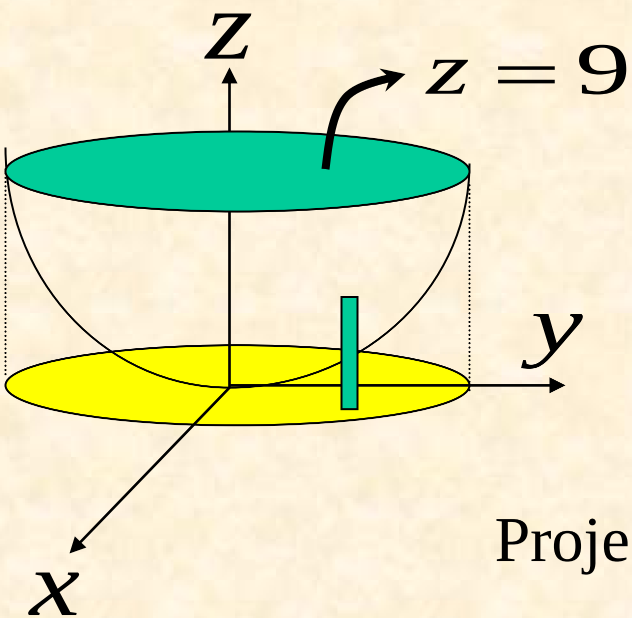
$$\iint_D (\sin x + \sin y)\,dA = \int_0^{\pi} \int_0^{\frac{\pi}{2}} (\sin x + \sin y)\,dy\,dx$$

$$= \int_0^{\pi} (y \sin x - \cos y) \Big|_0^{\frac{\pi}{2}} dx = \int_0^{\pi} (\frac{\pi}{2} \sin x + 1)\,dx$$

$$= -\frac{\pi}{2} \cos x + x \Big|_0^{\pi} = \frac{\pi^2}{2} + \pi + \frac{\pi}{2}$$

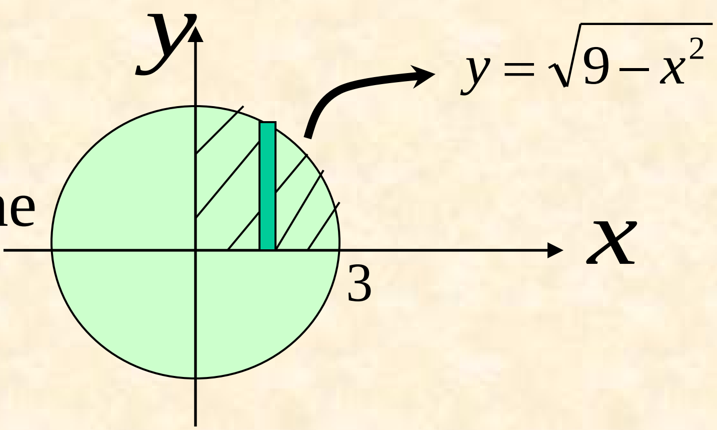
Exercises. **Find the volume under the surface** $z = x^2 + y^2$ **and bounded by the planes** $Z = 0$ *and* $Z = 9$

Solution. Projection in xy-plane, dydx:



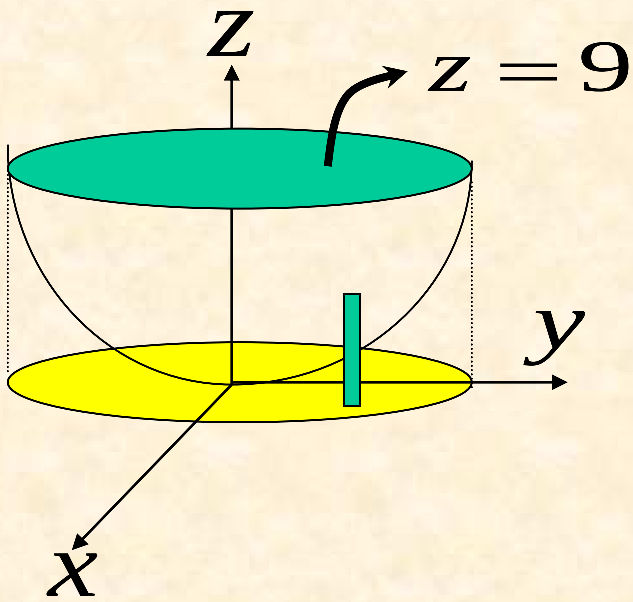
$$\begin{cases} z = 9 \\ z = x^2 + y^2 \end{cases} \Rightarrow x^2 + y^2 = 9$$

Projection in xy-plane

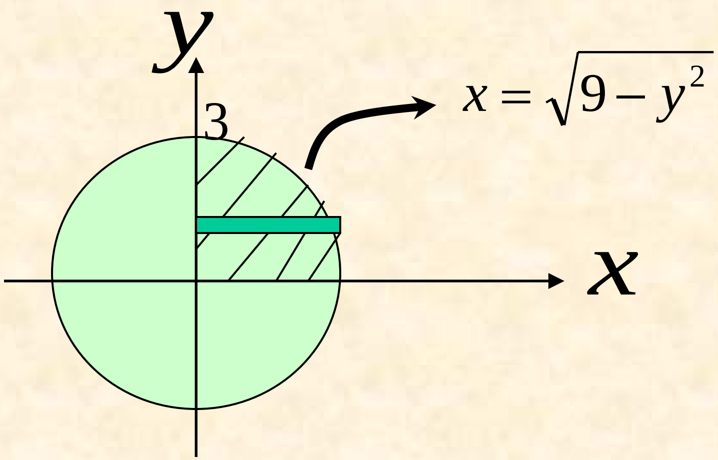


$$v = 4\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)\,dydx$$

Solution. Projection in xy-plane, dxdy:

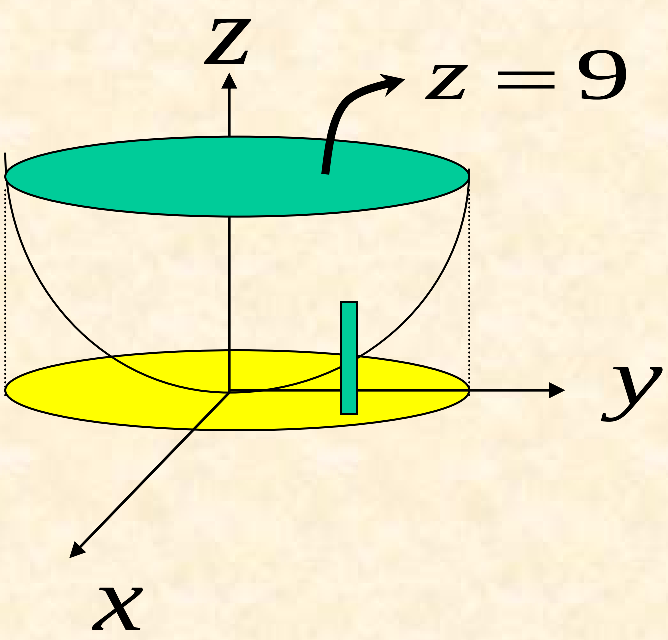


Projection in xy-plane



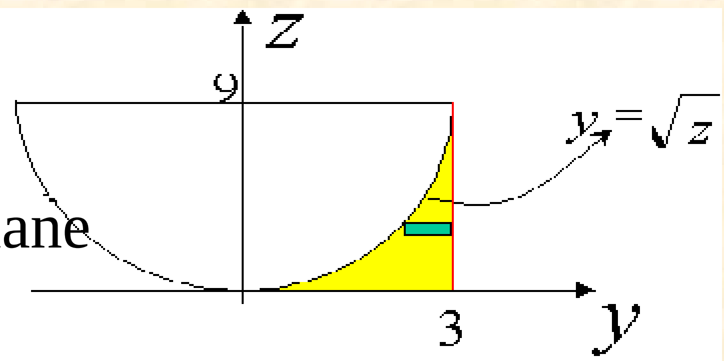
$$v = 4\int_0^3 \int_0^{\sqrt{9-y^2}} (x^2 + y^2)dxdy$$

Solution. Projection in yz-plane, dydz:



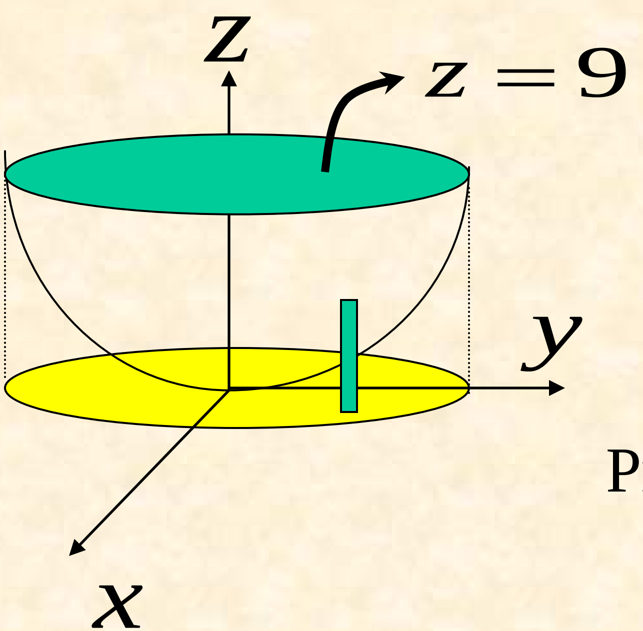
$$\begin{cases} x = 0 \\ z = x^2 + y^2 \end{cases} \Rightarrow z = y^2$$

Projection in yz-plane

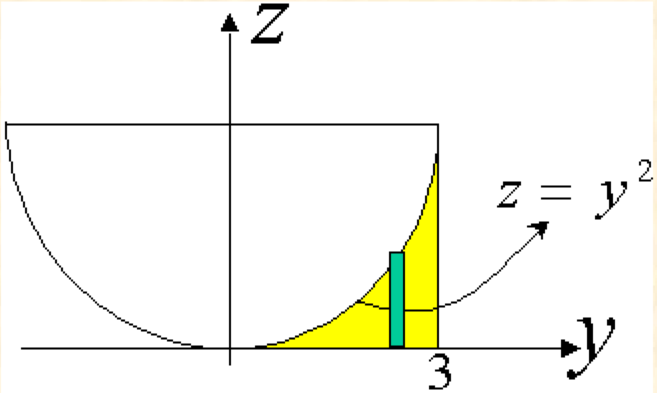


$$9 = x^2 + y^2 \Rightarrow x = \sqrt{9 - y^2} \Rightarrow \quad v = 2 \times 2 \int_0^9 \int_{\sqrt{z}}^3 \sqrt{9 - y^2}\, dydz$$

Solution. Projection in yz-plane, dzdy:



Projection in yz-plane

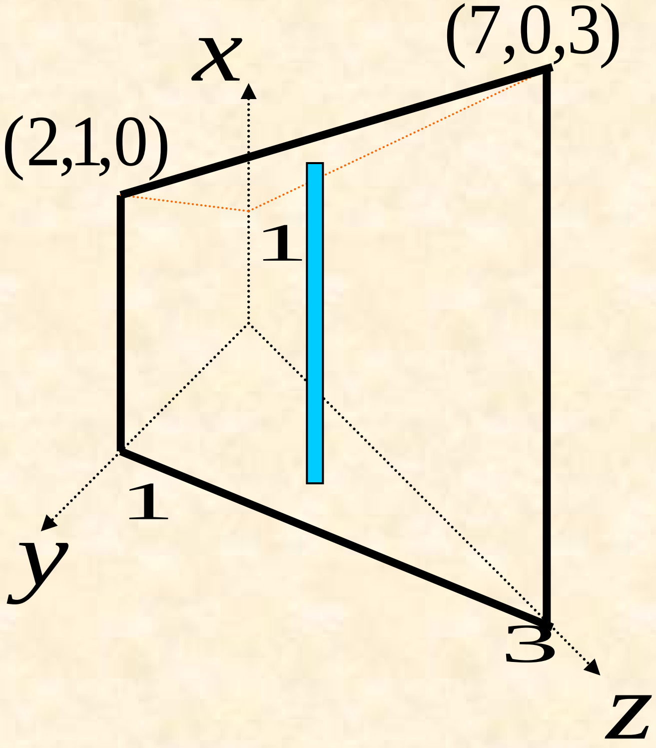


$$v = 2\times 2\int_0^3\int_0^{y^2}\sqrt{9-y^2}\,dzdy$$

Exercise. **Find the volume bounded by the planes**

$$x = y + 2z + 1, \quad 3y + z - 3 = 0, \quad x = 0, \quad y = 0, \quad z = 0$$

Solution.



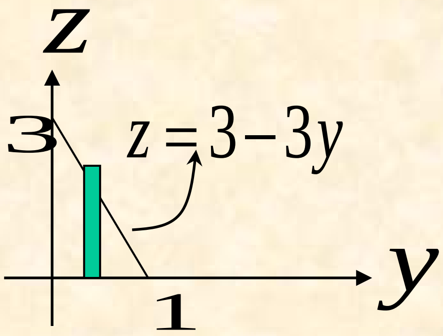
$$\begin{cases} z = 3 - 3y \\ z = 0 \end{cases} \Rightarrow y = 1 \qquad \begin{cases} z = 3 - 3y \\ y = 0 \end{cases} \Rightarrow z = 3$$

$$\begin{cases} x = y + 2z + 1 \\ y = 1, z = 0 \end{cases} \Rightarrow x = 2 \qquad \begin{cases} x = y + 2z + 1 \\ y = 0, z = 0 \end{cases} \Rightarrow x = 1$$

$$\begin{cases} x = y + 2z + 1 \\ y = 0, z = 3 \end{cases} \Rightarrow x = 7$$

Solution. Projection in yz-plane, dzdy:

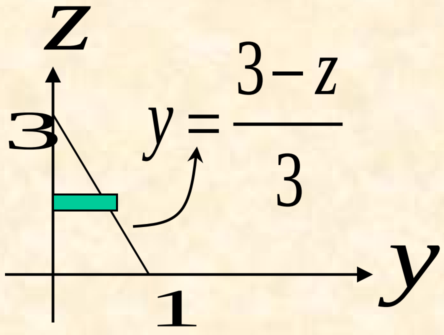
Projection in yz-plane



$$v=\int_0^1\int_0^{3-3y}(y+2z+1)\,dzdy$$

Projection in yz-plane, dydz:

تصویر در صفحه yz

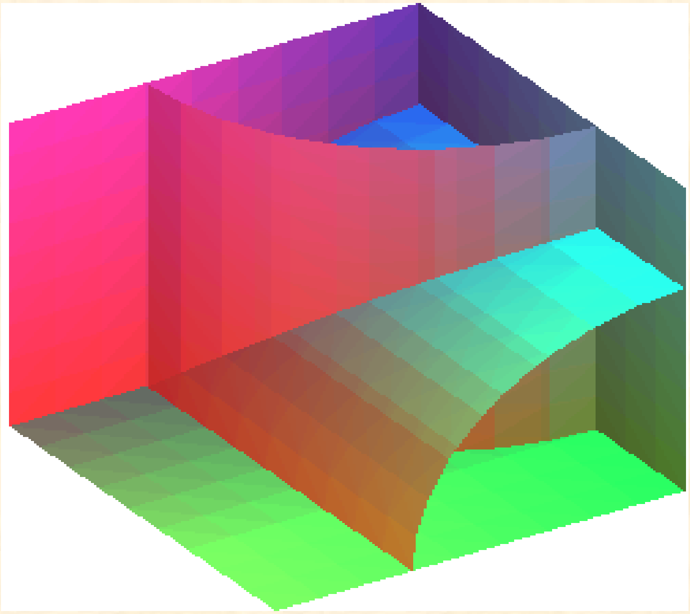


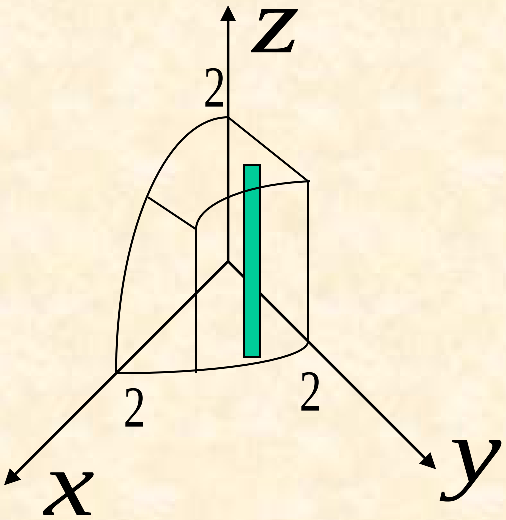
$$v=\int_0^3\int_0^{\frac{3-z}{3}}(y+2z+1)\,dydz$$

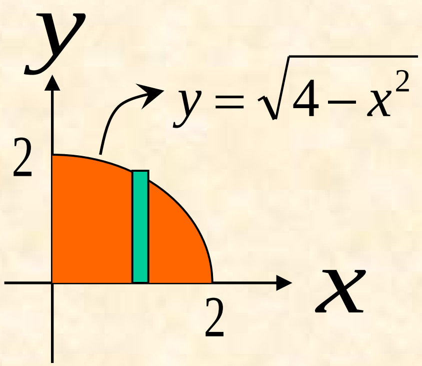
Exercise. **Find the volume in the first octant bounded by**

$$x^2 + y^2 = 4, \qquad x^2 + z^2 = 4$$

Solution.







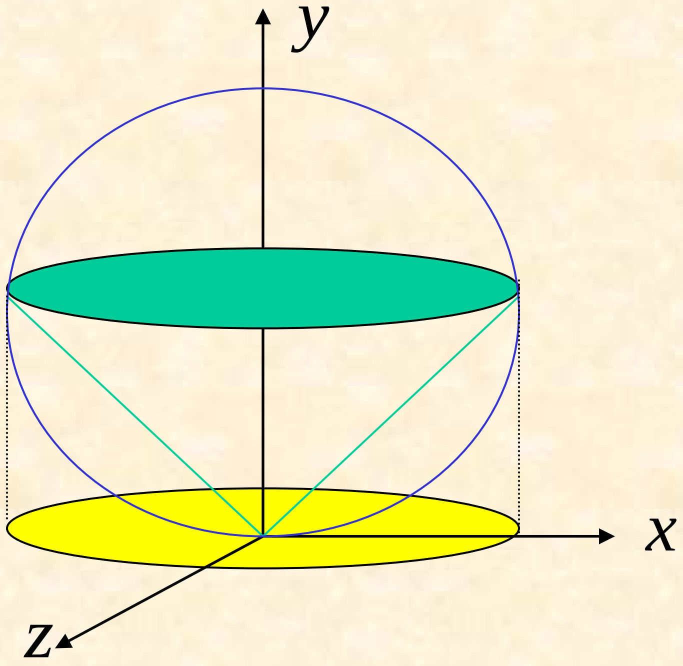
$$v = \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{4 - x^2}\,dydx = \int_0^2 (\sqrt{4-x^2}\,y\,|_0^{\sqrt{4-x^2}})dx = \int_0^2 (4 - x^2)dx$$

$$= (4x - \frac{x^3}{3})\,|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

Exercise. **Find the volume bounded by**

$$x^2 + z^2 = y^2, \quad y > 0 \text{ and } x^2 + (y-1)^2 + z^2 = 1$$

Solution.



$$\begin{cases} x^2 + (y-1)^2 + z^2 = 1 \\ x^2 + z^2 = y^2 \end{cases}$$

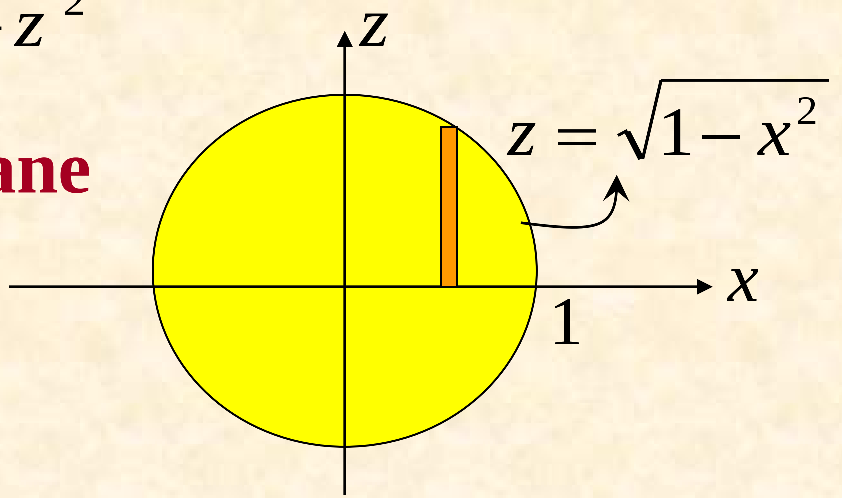
$$\Rightarrow 2y^2 - 2y = 0 \Rightarrow 2y(y-1) = 0 \Rightarrow$$

$y = 0, y = 1,$ $\qquad y = 1 \Rightarrow x^2 + z^2 = 1$ **a circle**

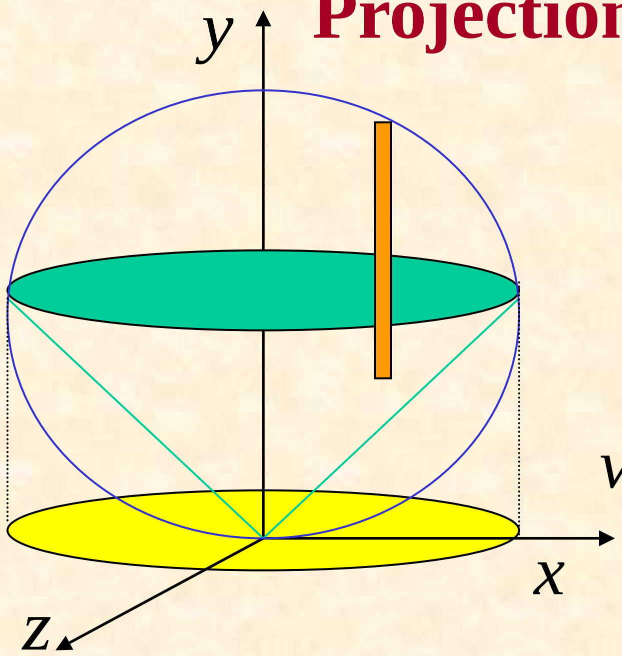
$y = 0 \Rightarrow x^2 + z^2 = 0 \Rightarrow (0,0,0)$ **The origin**

$$x^2 + (y-1)^2 + z^2 = 1, y \geq 1 \Rightarrow y = \sqrt{1 - x^2 - z^2} + 1$$

$$x^2 + z^2 = y^2, \quad y \geq 0 \Rightarrow y = \sqrt{x^2 + z^2}$$



Projection in the xz-plane

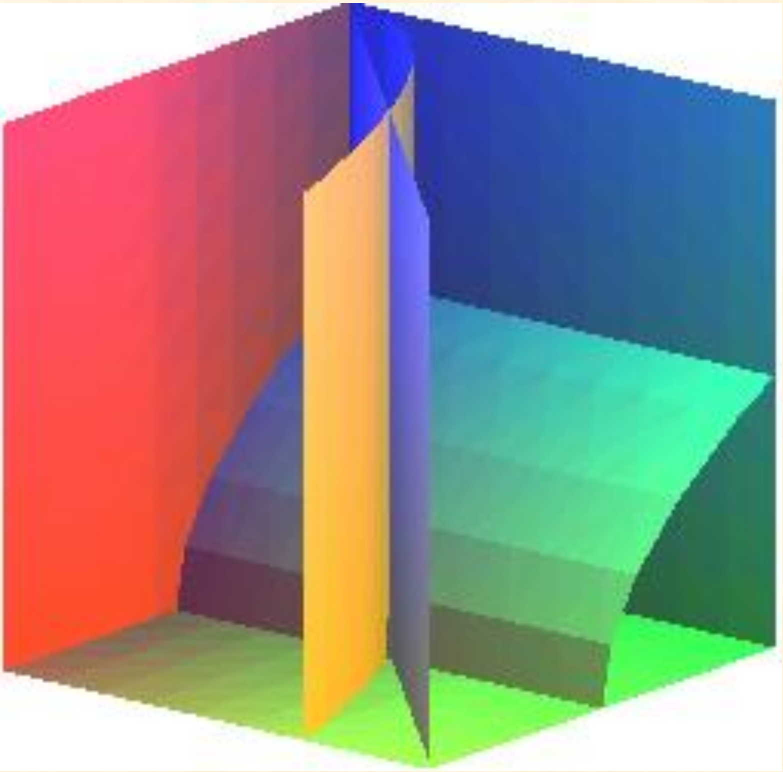


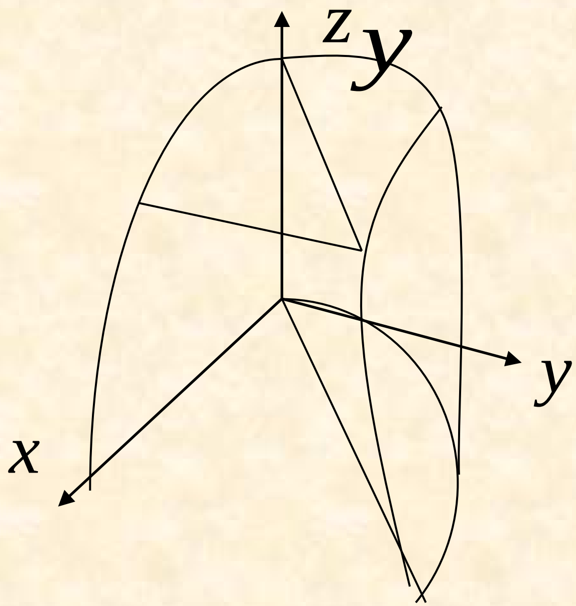
$$v = 4\int_0^1 \int_0^{\sqrt{1-x^2}} (\sqrt{1 - x^2 - z^2} + 1 - \sqrt{x^2 + z^2})\,dz\,dx$$

Exercise. **Find the volume in the first octant bounded by**

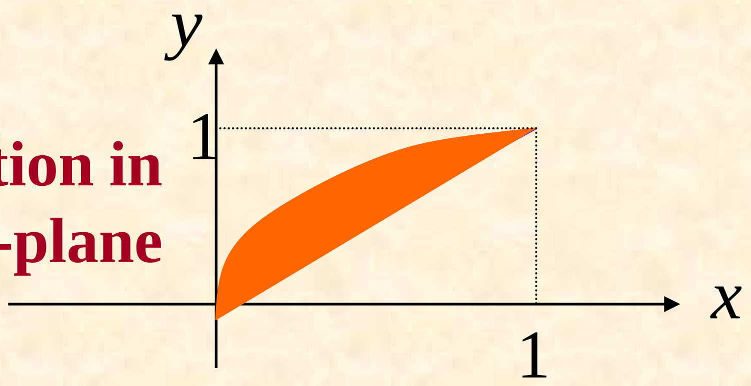
$$x^2 + z^2 = 1, \quad x = y^2, \quad x = y$$

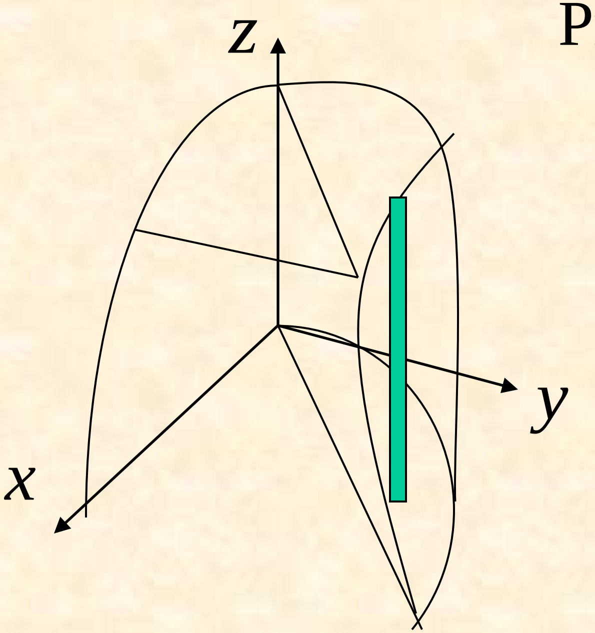
Solution.



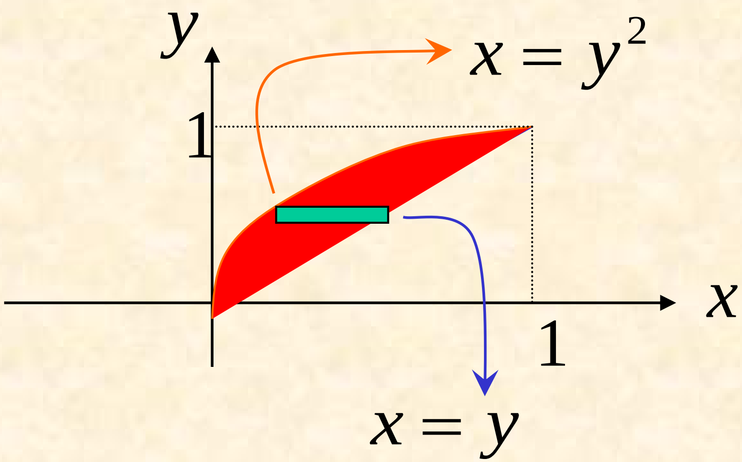


Projection in the xy-plane



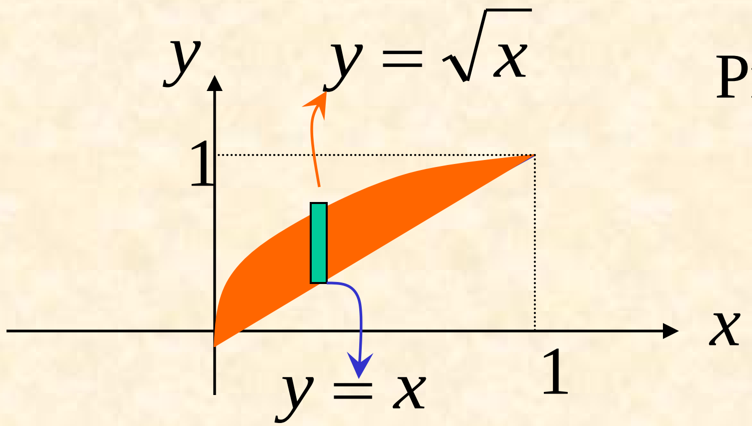


Projection in xy-plane, dxdy:



$$x^2 + z^2 = 1 \Rightarrow z = \sqrt{1 - x^2}$$

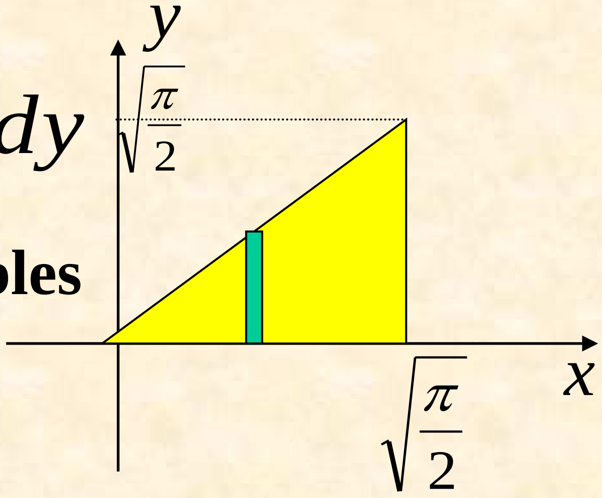
$$v = \int_0^1 \int_{y^2}^{y} \sqrt{1 - x^2}\, dx\, dy$$



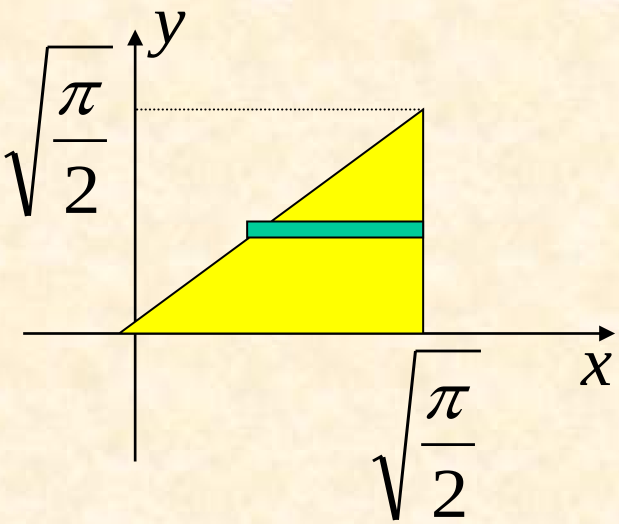
Projection in xy-plane, dydx:

$$v = \int_0^1 \int_{x}^{\sqrt{x}} \sqrt{1 - x^2}\, dy\, dx$$

Exercise. **Find** $I = \int_0^{\sqrt{\frac{\pi}{2}}} \int_y^{\sqrt{\frac{\pi}{2}}} \sin x^2 dxdy$



Solution. **We should change the order of variables**



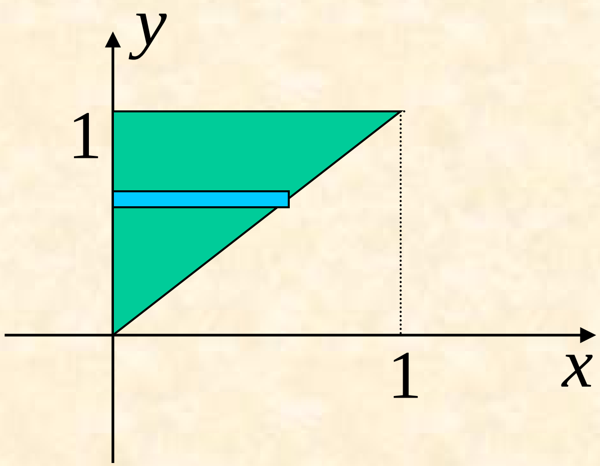
$$I = \int_0^{\sqrt{\frac{\pi}{2}}} \int_0^x \sin x^2 dydx = \int_0^{\sqrt{\frac{\pi}{2}}} (y \sin x^2 |_0^x) dx$$

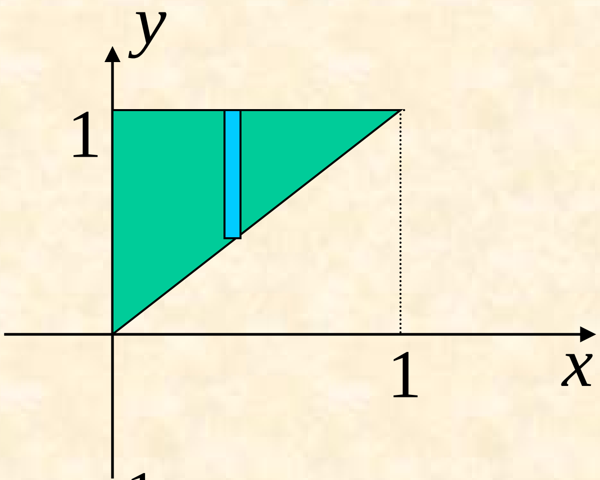
$$= \int_0^{\sqrt{\frac{\pi}{2}}} x \sin x^2 dx = -\frac{1}{2} \cos x^2 |_0^{\sqrt{\frac{\pi}{2}}}$$

$$= -\frac{1}{2}(\cos\frac{\pi}{2} - \cos 0) = \frac{1}{2}$$

Exercise. **Find** $$I = \int_0^1 \int_x^1 e^{y^2}\,dy\,dx$$

Solution. **We should change the order of variables**



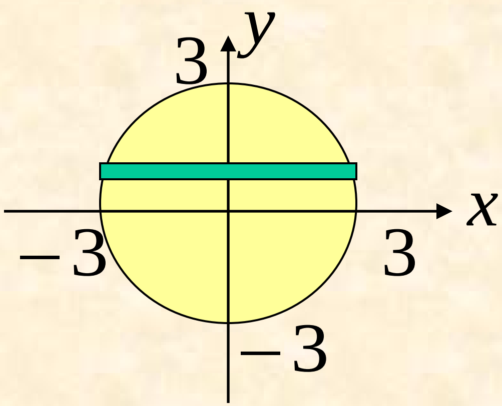


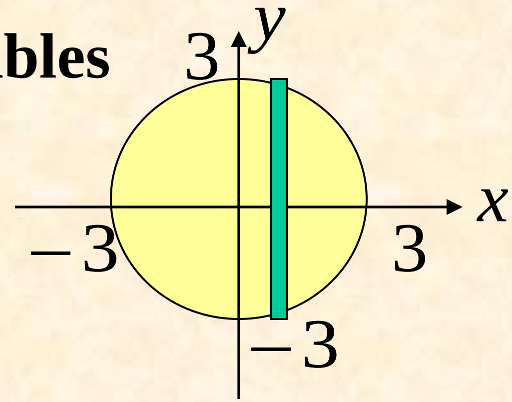
$$I = \int_0^1 \int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left(x e^{y^2} \Big|_0^y\right) dy = \int_0^1 y e^{y^2}\,dy$$

$$= \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{1}{2} e - \frac{1}{2}$$

Exercise. **Find** $$I=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2\sqrt{9-y^2}\,dydx$$

Solution. **We should change the order of variables**





$$I=\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2\sqrt{9-y^2}\,dxdy$$

$$=\int_{-3}^{3}\left(\frac{x^3}{3}\sqrt{9-y^2}\,\Big|_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\right)dy=\int_{-3}^{3}\frac{2}{3}(9-y^2)^2\,dy$$

$$=\int_{-3}^{3}\frac{2}{3}(81+y^4-18y)\;dy=\frac{2}{3}(81y+\frac{y^5}{5}-9y^2)\,\Big|_{-3}^{3}$$