Sigma $\left(\sum\right)$ Notation

$$\sum_{i=n}^{m} a_i = a_n + a_{n+1} + \ldots + a_m$$

$$\sum_{i=n}^{m} f(i) = f(n) + f(n+1) + f(n+2) + \ldots + f(m)$$

Sigma Properties

1) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

3) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$

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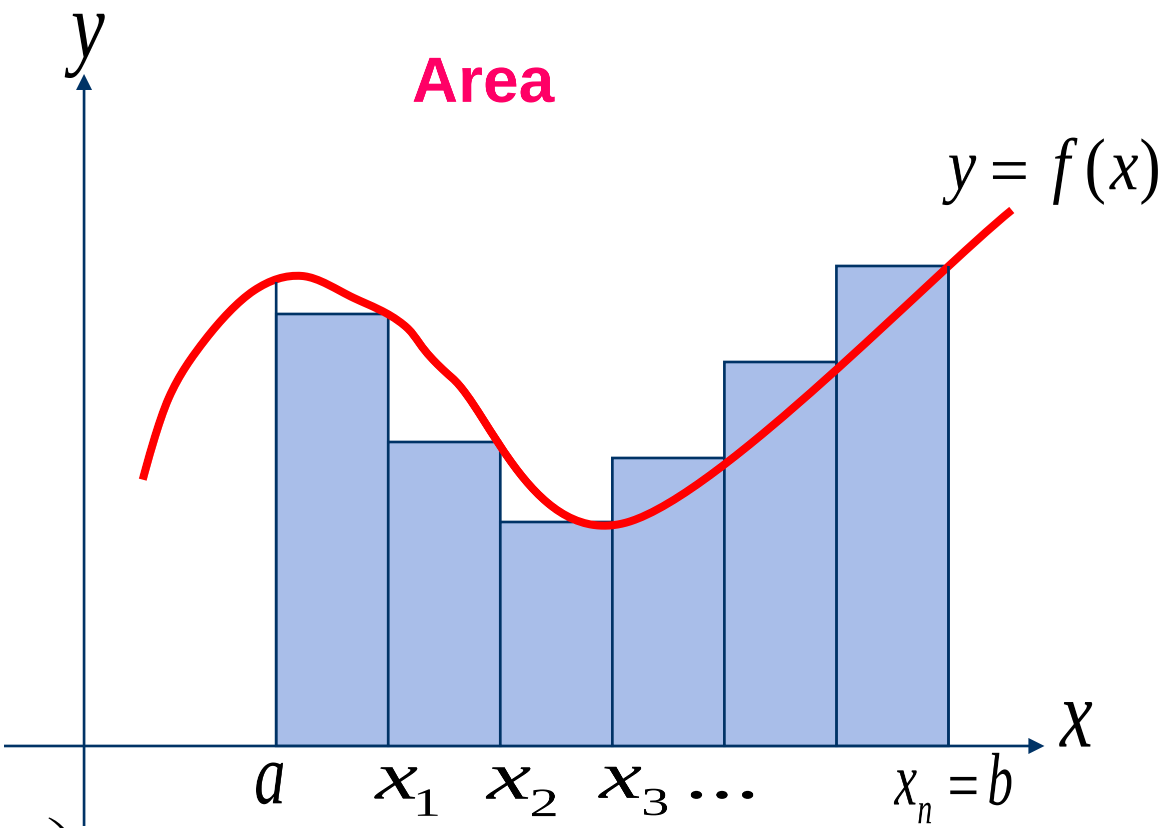
$$4)\sum_{i=1}^{n} i^4 = \frac{n(n+1)\left(6n^3+9n^2+n-1\right)}{30}$$

$$5)\sum_{k=n}^{m} (ca_k) = c\sum_{k=n}^{m} a_k$$

$$6)\sum_{i=n}^{m} (a_i \pm b_i) = \sum_{i=n}^{m} a_i \pm \sum_{i=n}^{m} b_i$$

$$7)\sum_{i=n}^{m} c = (m-n+1)c$$



$$S = \lim_{n \to +\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right) . \frac{b-a}{n}$$

$$\Rightarrow S = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)$$

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Exercise. **Find the area bounded by**

$$y = x^2 + 1, \ \ x = 0, \ \ x = 1, \ \ y = 0.$$

Solution.

$$S = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f\left(0 + \frac{1}{n}\right) = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i^2}{n^2} + 1\right) = \lim_{n\to+\infty} \frac{1}{n}\left(\sum_{i=1}^{n}\frac{i^2}{n^2} + \sum_{i=1}^{n} 1\right)$$

$$\lim_{n\to+\infty} \frac{1}{n}\left(\frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + n\right) = \lim_{n\to+\infty}\left(\frac{2n^3 + \ldots}{6n^3} + 1\right) = \frac{1}{3} + 1 = \frac{4}{3}$$

Definition of Definite Integral. (a) If $a \le b$, then

$$\int_a^b f(x)dx = \lim_{n\to+\infty} \frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{b-a}{n}i\right)$$

(b) If $a > b$, then $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Exercise. $\int_2^{-1}\left(x^3 - x\right)dx = ?$

Solution. $\int_{-1}^{2}\left(x^3 - x\right)dx = \lim_{n\to+\infty} \frac{3}{n}\sum_{i=1}^{n}\left(\frac{3i}{n}-1\right)^3 - \left(-1+\frac{3i}{n}\right)$

$$= \lim_{n\to+\infty} \frac{3}{n}\sum_{i=1}^{n}\left(\frac{27i^3}{n^3}-\frac{27i^2}{n^2}+\frac{6i}{n}\right)$$

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$$\lim_{n\to+\infty}\frac{3}{n}\left(\frac{27}{n^3}\sum i^3-\frac{27}{n^2}\sum i^2+\frac{6}{n}\sum i\right)=$$

$$\lim_{n\to+\infty}\frac{81}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)-\frac{81}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$+\frac{18}{n^2}\left(\frac{n(n+1)}{2}\right)=\frac{81}{4}-27+9=\frac{9}{4}$$

$$\Rightarrow\int_2^{-1}\left(x^3-x\right)dx=-\frac{9}{4}$$

Definite Integral Properties

1) $\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

2) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

3) $\int_a^b c\,dx = c(b-a)$

4) $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

5) If $f(x) \geq g(x)$, $x \in [a,b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

Exercise. $$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Solution.

$$\text{Clearly } -|f(x)| \le f(x) \le |f(x)|$$

$$\Rightarrow -\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

$$-c \le u \le c \Rightarrow |u| \le c \quad \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Exercise. Show that $0 \le \int_0^1 \sin x^2 dx \le \frac{1}{3}$

Solution.

$$0 \le u \Rightarrow \sin u \le u$$

$$\Rightarrow 0 \le \sin x^2 \le x^2 \Rightarrow \int_0^1 0 dx \le \int_0^1 \sin x^2 dx \le \int_0^1 x^2$$

Now by the definition of the definite integral, $\int_0^1 x^2 = \frac{1}{3}$

$$\Rightarrow 0 \le \int_0^{\frac{\pi}{2}} \sin x^2 dx \le \frac{1}{3}$$

Exercise. $\int_{-1}^{1}[2x]dx = ?$

Solution.

$$=\int_{-1}^{\frac{-1}{2}}[2x]dx+\int_{\frac{-1}{2}}^{0}[2x]dx+\int_{0}^{\frac{1}{2}}[2x]dx+\int_{\frac{1}{2}}^{1}[2x]dx$$

$$-1<x<\frac{-1}{2}\Rightarrow -2<2x<-1\Rightarrow [2x]=-2$$

$$=\int_{-1}^{\frac{-1}{2}}-2dx+\int_{\frac{-1}{2}}^{0}-1dx+\int_{0}^{\frac{1}{2}}0dx+\int_{\frac{1}{2}}^{1}1dx$$

$$=-2(\frac{-1}{2}+1)-(0+\frac{1}{2})+(1-\frac{1}{2})$$

First Fundamental Theorem of Calculus

If the function f is continious on [a,b], and

and $g(x) = \int_a^x f(t)dt \quad a \le x \le b$, then the

function g is differentiable of [a,b] and

$$g'(x) = f(x)$$

Corollary. If $g(x) = \int_{h(x)}^{k(x)} f(t)dt$, then

$$g(x) = f(k(x)) \times k'(x) - f(h(x)) \times h'(x)$$

Exercise. $$F(x)=\int_{\sin x^3}^{\tan^2(x^4)}\sqrt{1+t^4}\,dt \qquad Find\ \ F'(x)$$

Solution.

$$F'(x)=\sqrt{1+\tan^8(x^4)}\times 2\tan(x^4)\times\sec^2(x^4)\times 4x^3$$

$$-\sqrt{1+\sin^4(x^3)}\times\cos(x^3)\times 3x^2$$

Exercise. **Show that** $x \succ 0 \ , \ g(x) = \int_{x}^{2x} \frac{dt}{t}$

Solution.

$$g(x) = \int_{x}^{2x} \frac{dt}{t} \ \Rightarrow \ g'(x) = 2 \times \frac{1}{2x} - \frac{1}{x} = 0 \ \Rightarrow$$

g is a constant function.

Corollary. If $k(x)=\int_{g(x)}^{h(x)} f(x,t)\,dt$, then

$$k'(x)=f\left(x,h(x)\right)\times h'(x)-f\left(x,g(x)\right)\times g'(x)$$

$$+\int_{g(x)}^{h(x)} f_x(x,t)\,dt$$

Exercise. $$y=\int_0^x \cos(x^2\sqrt[3]{t})\,dt, \quad y'=?$$

Solution.

$$x^2\sqrt[3]{t}=\sqrt[3]{x^6t} \Rightarrow x^6t=u \Rightarrow x^6\,dt=du \Rightarrow dt=\frac{du}{x^6}$$

$$=\int_0^{x^7}\cos\sqrt[3]{u}\,\frac{du}{x^6}=\frac{1}{x^6}\int_0^{x^7}\cos\sqrt[3]{u}\,du=x^{-6}\times\int_0^{x^7}\cos\sqrt[3]{u}\,du$$

$$\Rightarrow f'(x)=-6x^{-7}\times\int_0^{x^7}\cos\sqrt[3]{u}\,du+x^{-6}\left(7x^6\cos\sqrt[3]{x^7}\right)$$

Second Fundamental Theorem of Calculus

If the function f is continious on $[a,b]$, and and g is a function such that

$$\forall x \in [a,b] : f(x) = g'(x), \text{ then}$$

$$\int_a^b f(x)dx = g(b) - g(a).$$

Exercise.

$$\int_0^{\pi}\left|\cos x+\frac{1}{2}\right|dx=?$$

Solution.

$$\int_0^{\pi}\left|\cos x+\frac{1}{2}\right|dx=\int_0^{\frac{2\pi}{3}}(\cos x+\frac{1}{2})dx+\int_{\frac{2\pi}{3}}^{\pi}-(\cos x+\frac{1}{2})dx$$

$$=(\sin x+\frac{x}{2})\Big|_0^{\frac{2\pi}{3}}+(-\sin x-\frac{x}{2})\Big|_{\frac{2\pi}{3}}^{\pi}=$$

$$\frac{\sqrt{3}}{2}+\frac{\pi}{3}-(-0-\frac{\sqrt{3}}{2}+\frac{\pi}{2}-\frac{\pi}{3})=\sqrt{3}+\frac{\pi}{6}$$

Exercise. Find

$$\lim_{n\to+\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{\frac{3}{2}}}$$

Solution.

$$=\lim_{n\to+\infty}\frac{1}{n\sqrt{n}}\sum_{i=1}^{n}\sqrt{i}=\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{i}{n}}=$$

$$\int_0^1\sqrt{x}\,dx=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1=\frac{2}{3}$$

$$\lim_{x \to 0} \frac{1}{x} \int_0^x \sin\sqrt{xt}\,dt = ?$$

$$\begin{cases} x > 0 \Rightarrow 0 \le t \le x \\ x < 0 \Rightarrow x \le t \le 0 \end{cases} \Rightarrow 0 \le xt \Rightarrow 0 \le \sin\sqrt{xt} \le \sqrt{xt}$$

$$\Rightarrow 0 \le \int_0^x \sin\sqrt{xt}\,dt \le \int_0^x \sqrt{xt}\,dt = \sqrt{x}\int_0^x t^{\frac{1}{2}}dt = \sqrt{x} \times \frac{2}{3} t^{\frac{3}{2}} \Big|_0^x = \frac{2}{3}x^2$$

$$\Rightarrow 0 \le \frac{1}{x}\int_0^x \sin\sqrt{xt}\,dt \le \frac{2}{3}x \qquad \lim_{x\to 0} 0 = \lim_{x\to 0} \frac{2}{3}x = 0$$

Now by squeeze theorem $$\lim_{x\to 0} \frac{1}{x}\int_0^x \sin\sqrt{xt}\,dt = 0$$