Cylindrical & Spherical Coordinates

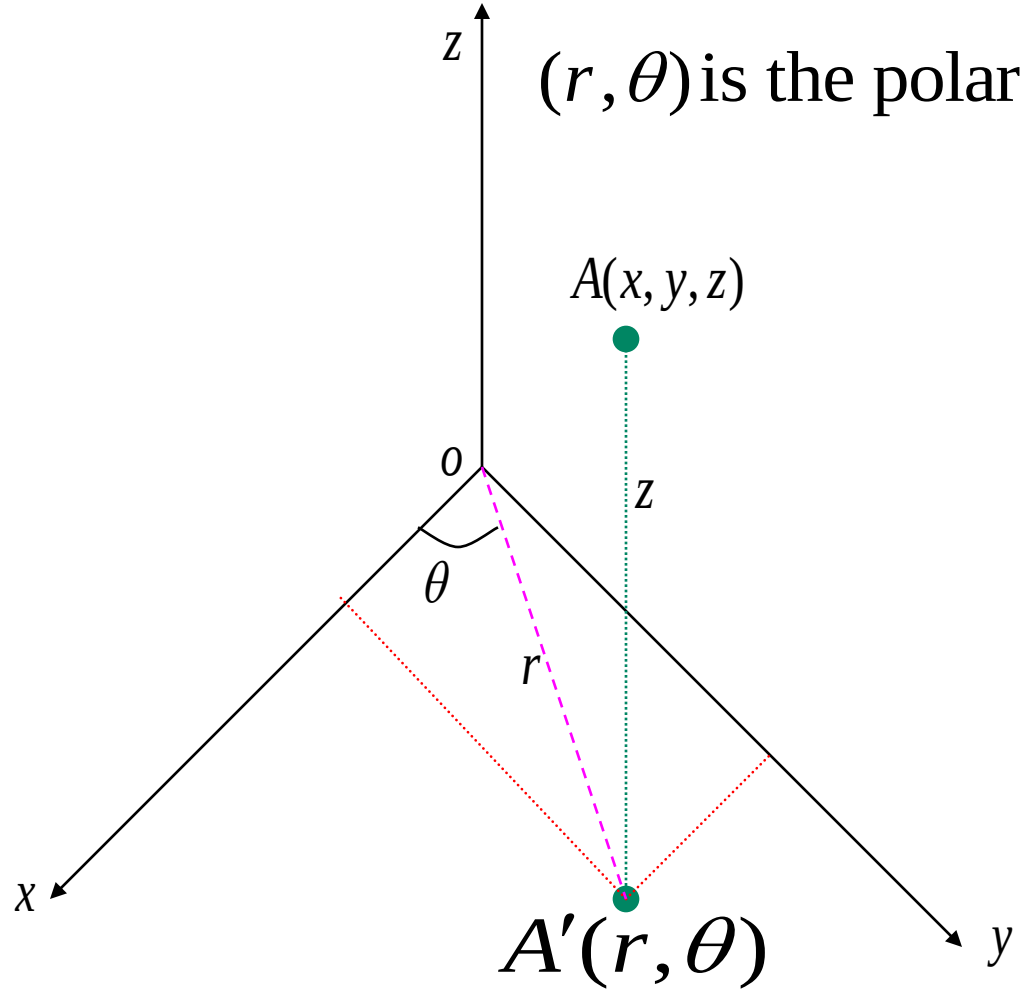
Cartesian Coordinates:

$A(x, y, z)$

Cylindrical Coordinates:

$A(r, \theta, z)$

The point A' is the projection of the point A in xy-plane.



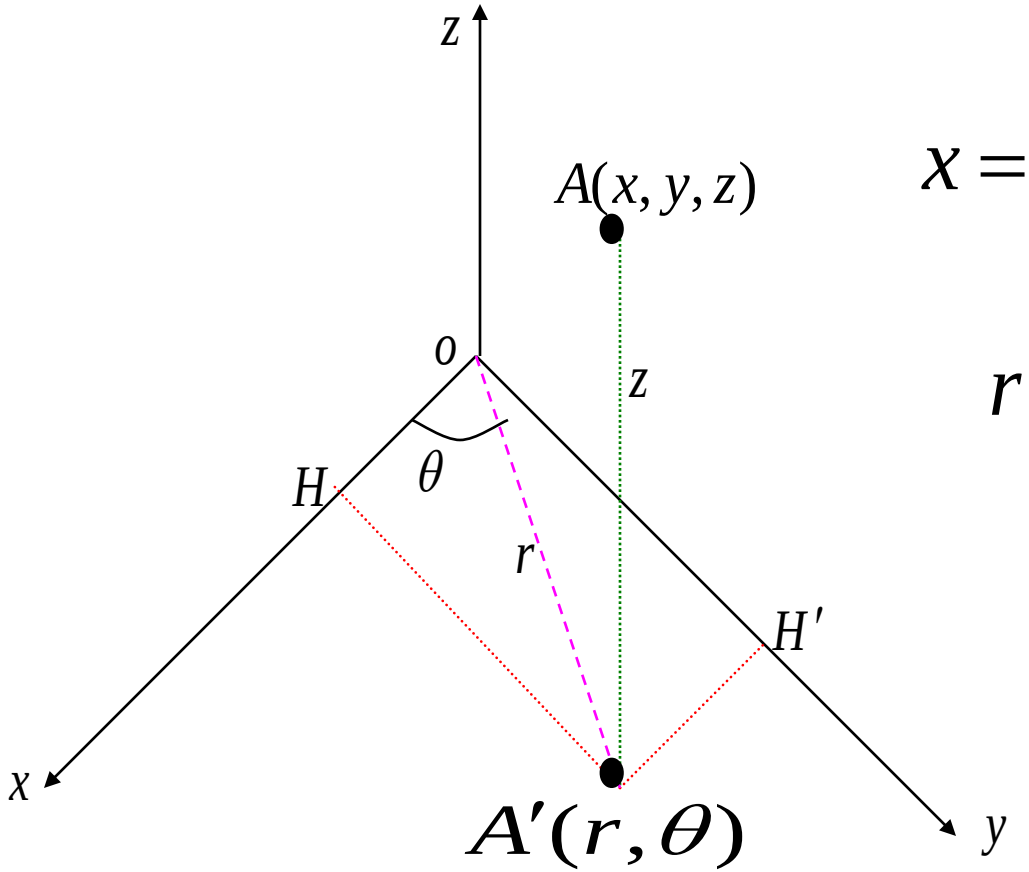
(r, θ) is the polar coordinates of A', where $r \geq 0$.

$$z = AA'$$

$$0 \leq r < +\infty$$

$$-\infty < \theta < +\infty$$

$$-\infty < z < +\infty$$



$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

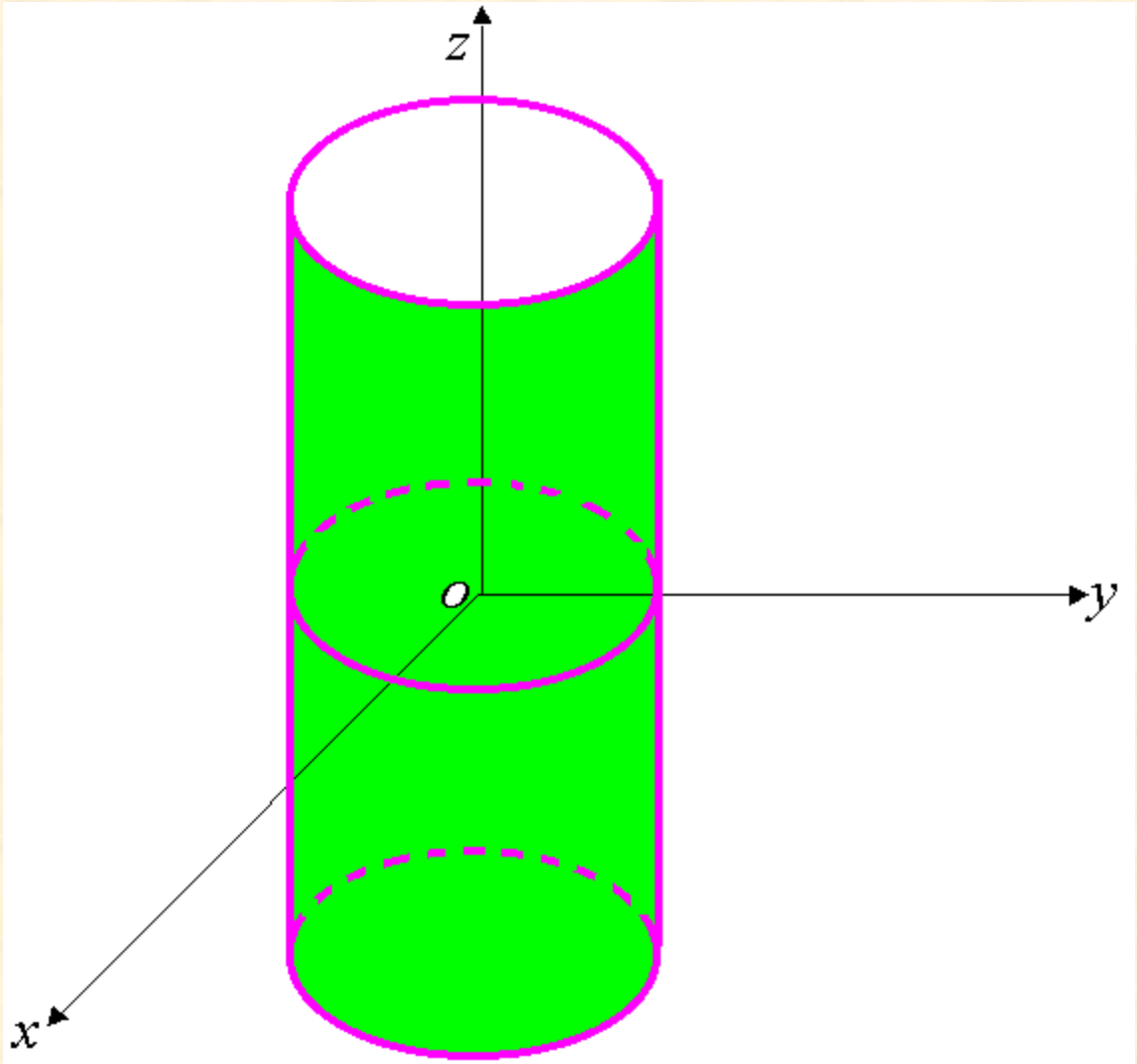
$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

Exercise. **Draw the following surfaces, where c is a constant.**

$(a)\ \ r = c \qquad (b)\ \ \theta = c \qquad (c)\ z = c$

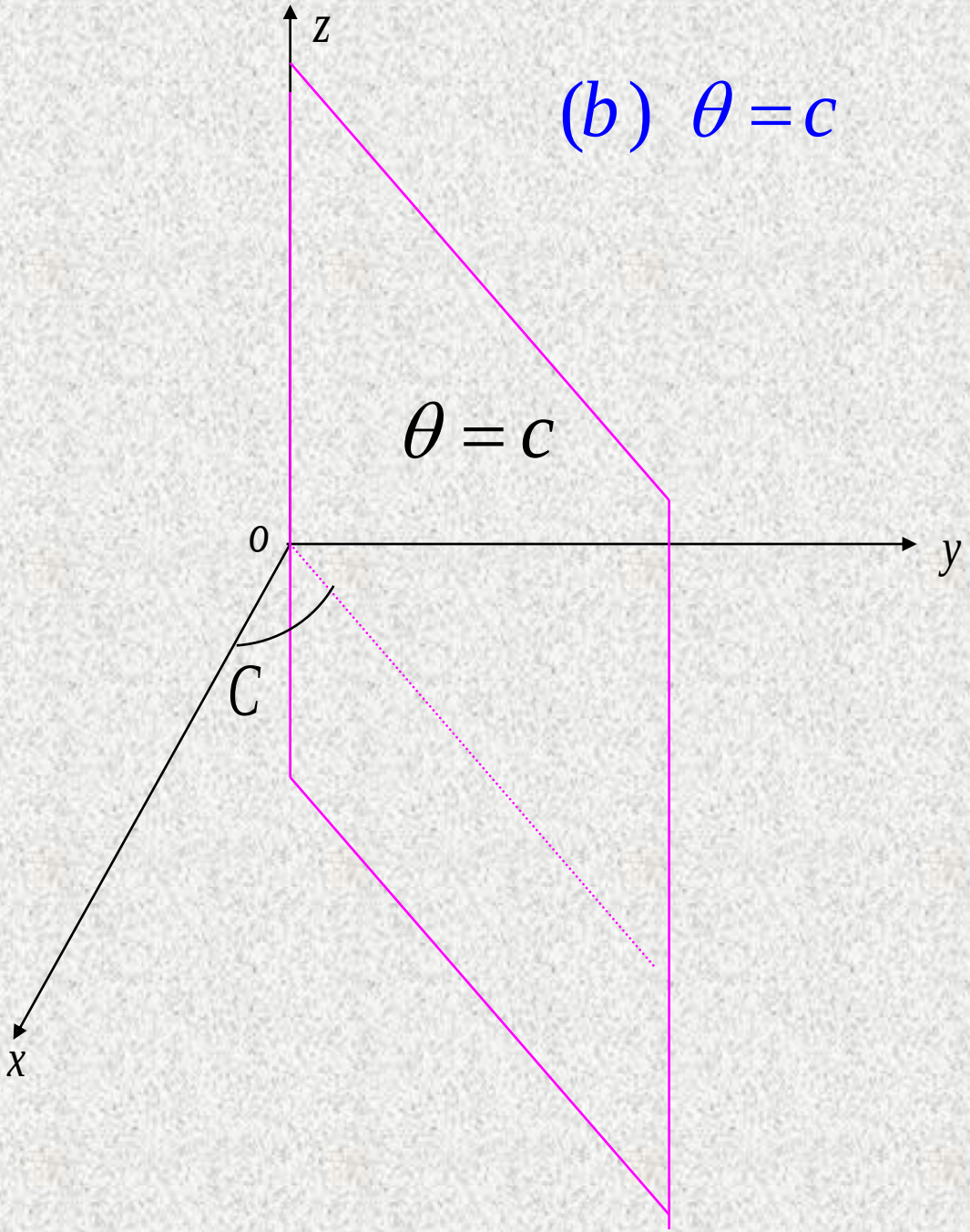
Solution.

$(a)\quad r = c$

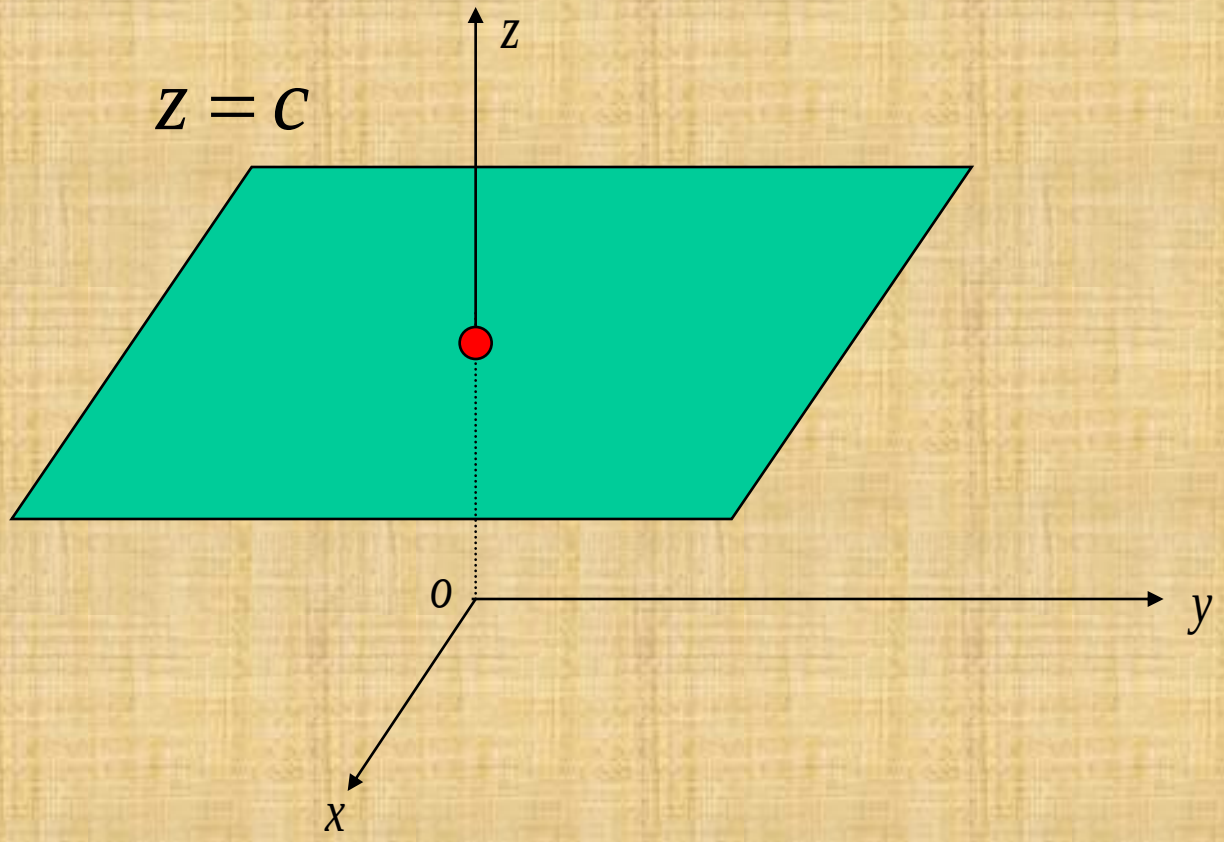


Solution.

$(b)\ \theta = c$

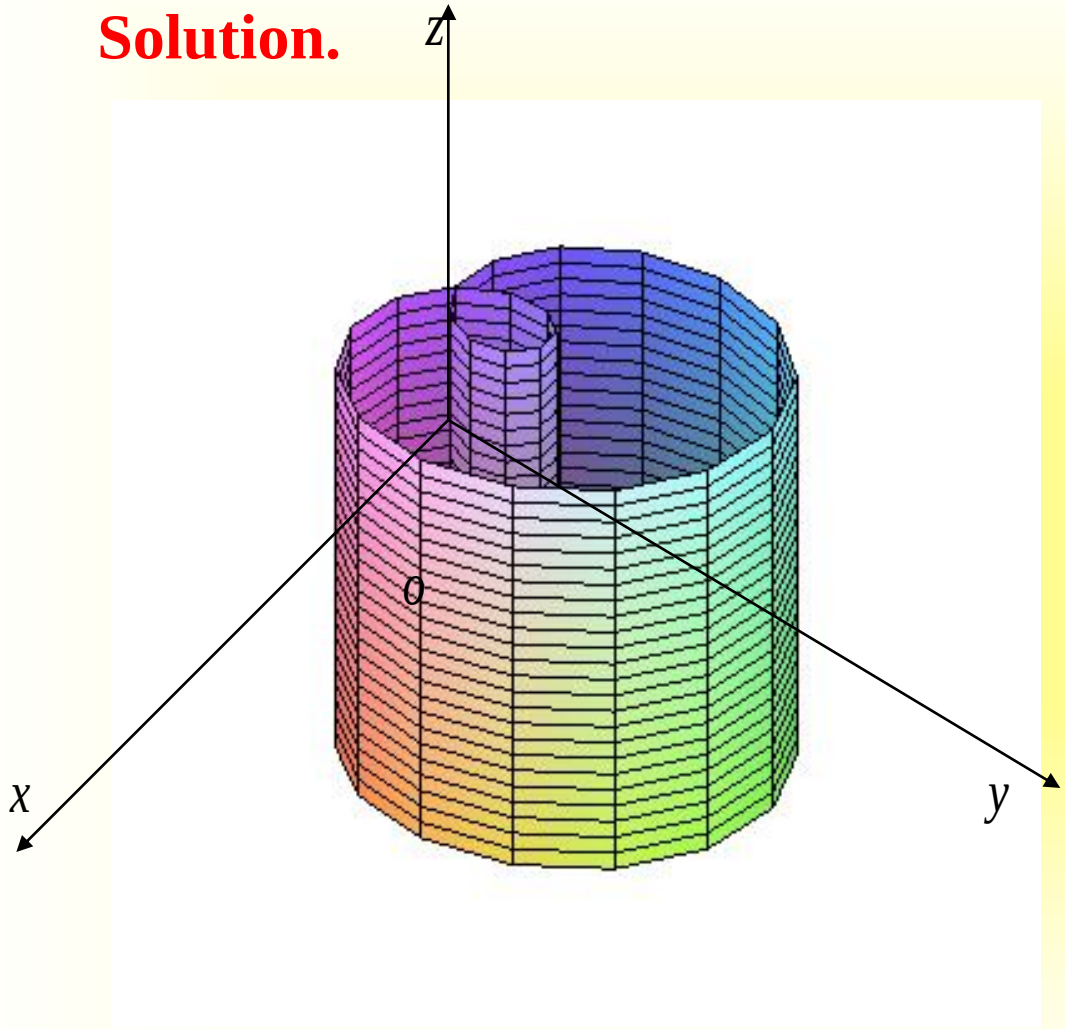


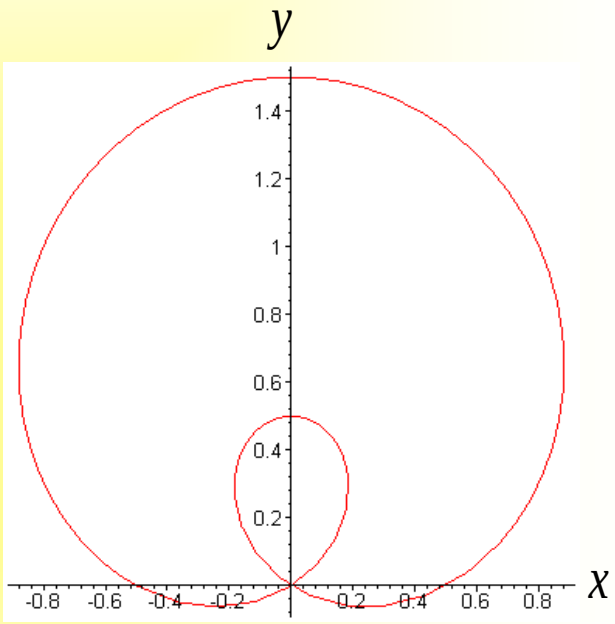
Solution.



Exercise. **Draw** $r = \frac{1}{2} + \sin\theta$

Solution.



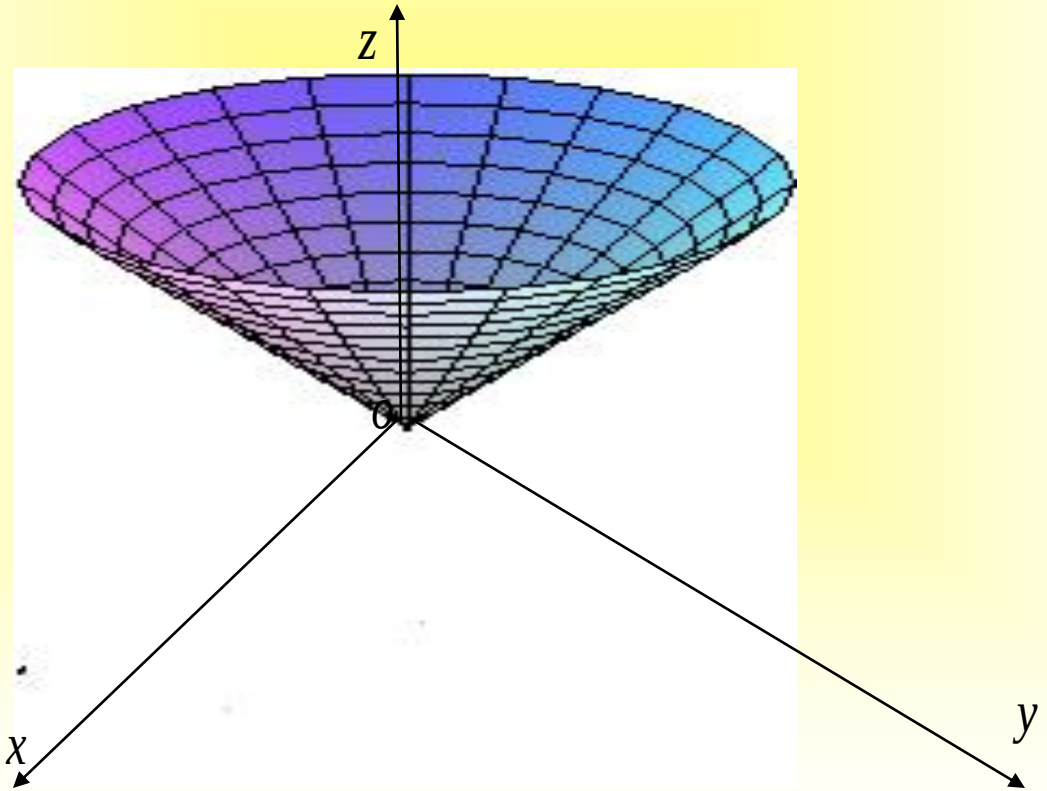


Exercise. **Draw r =z**

Solution.

$$r = z \Rightarrow r^2 = z^2,\ z \geq 0 \Rightarrow x^2 + y^2 = z^2,\ z \geq 0$$

So this surface is given by rotating the line y=z about the z-axis



Exercise. Change the cartisian equation $x^2 - y^2 = z$ to cylindrical coordinates.

Solution.

$$x^2 - y^2 = z \Rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = z$$

$$\Rightarrow r^2(\cos^2\theta - \sin^2\theta) = z \Rightarrow r^2\cos 2\theta = z$$

Cartesian Coordinates:

$A(x, y, z)$

Spherical Coordinates:

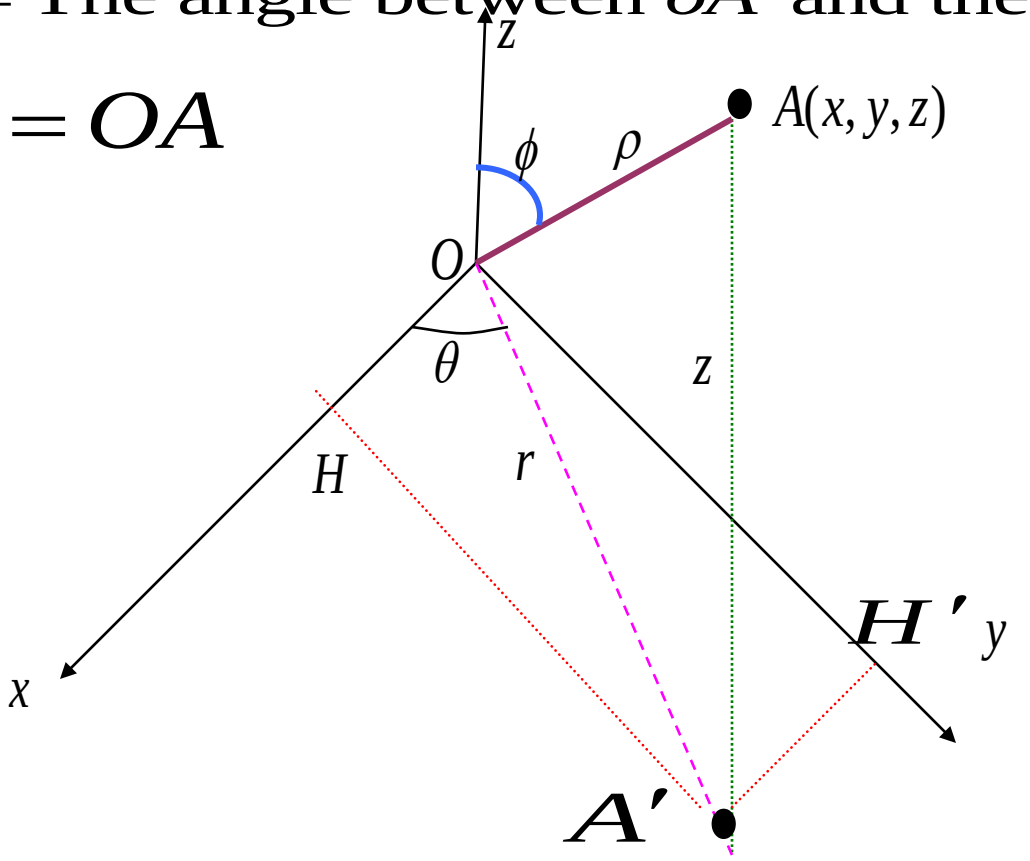
(ρ, θ, ϕ)

The point A' is the projection of the point A in xy-plane.

$\theta =$ The angle between oA' and the positive side of x-axis.

$\varphi =$ The angle between oA and the positive side of z-axis.

$\rho = OA$



$$0 \le \phi \le \pi$$

$$0 \le \rho < +\infty$$

$$-\infty < \theta < +\infty$$

$$r = \rho \sin\phi$$

$$\left.\begin{aligned} x &= r.\cos\theta \\ r &= \rho\sin\phi \end{aligned}\right\} \Rightarrow x = \rho\sin\phi\cos\theta$$

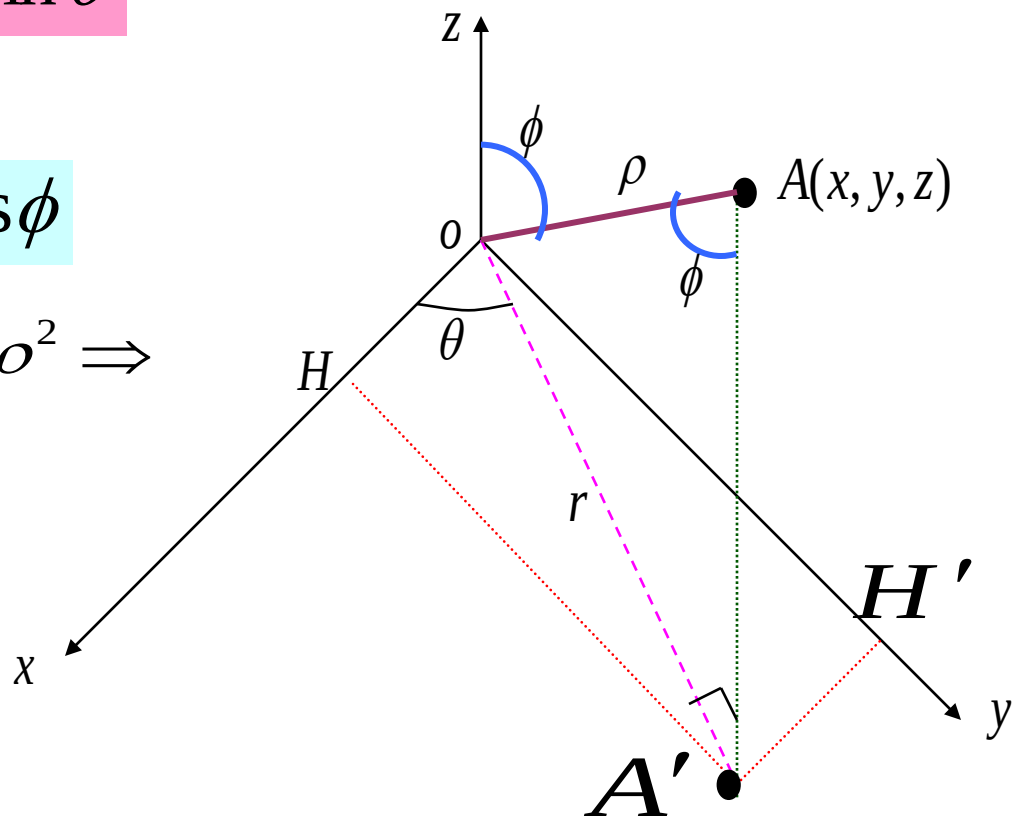
$$\left.\begin{aligned} y &= r.\sin\theta \\ r &= \rho\sin\phi \end{aligned}\right\} \Rightarrow y = \rho\sin\phi\sin\theta$$

$$z = AA' = \rho.\cos\phi \Rightarrow z = \rho\cos\phi$$

$$x^2 + y^2 + z^2 = r^2 + AA'^2 = \rho^2 \Rightarrow$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$r = \rho\sin\varphi$$



Exercise. **Draw the following surfaces, where c is a constant.**

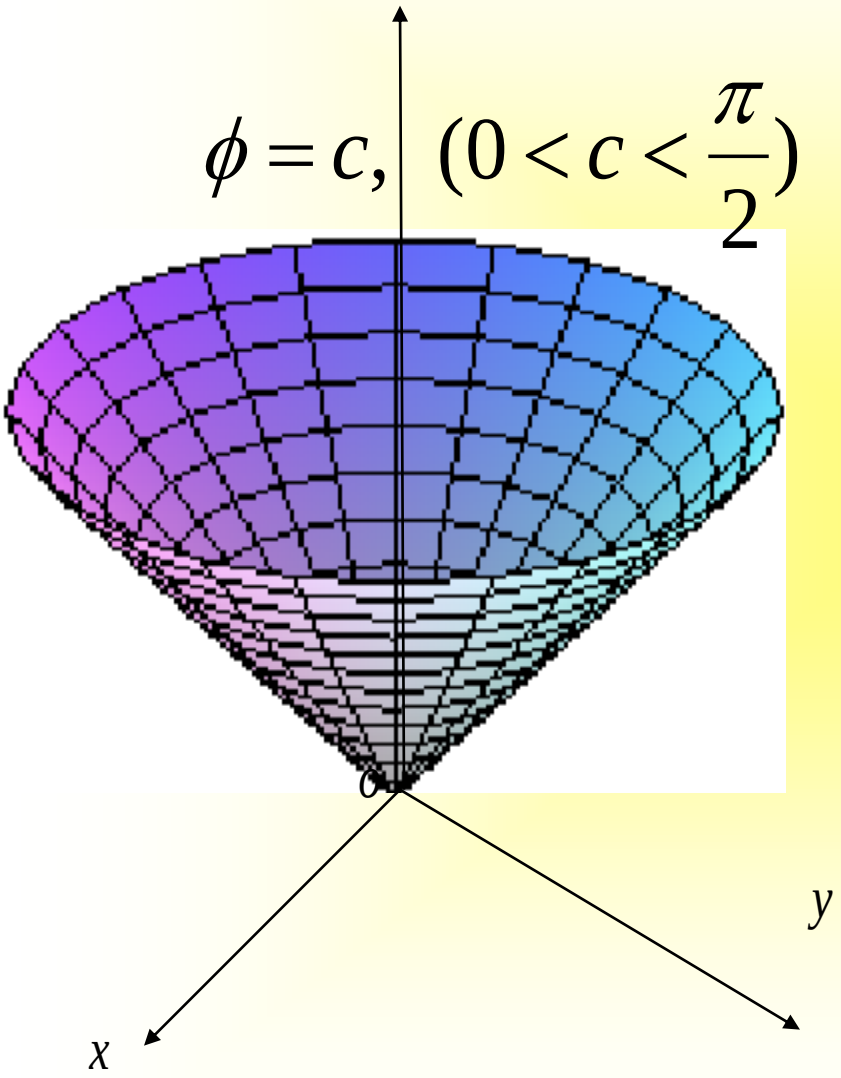
$(a)\ \varphi = c,\ (0 \le c \le \pi)$ $(b)\ \theta = c$ $(c)\ \rho = c,\ (c > 0)$

Solution.

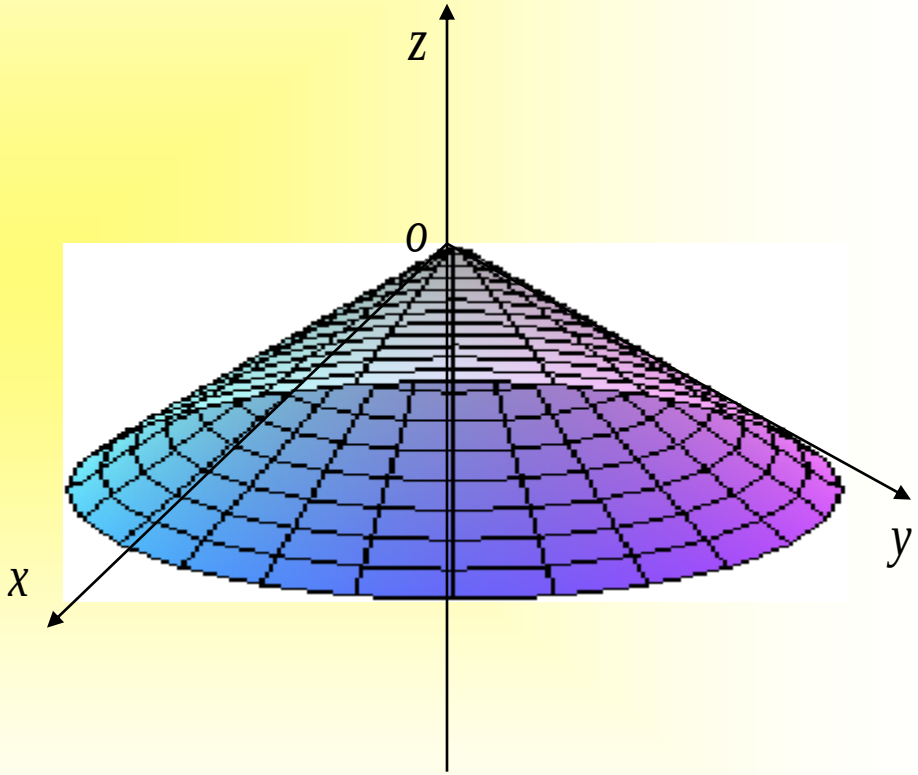
$(a)\ \varphi = c,\ \ (0 \le c \le \pi)$

a cone if $0 \ne c,\ \ c \ne \dfrac{\pi}{2},\ \ c \ne \pi$

$\phi = c,\ \ (0 < c < \dfrac{\pi}{2})$



$\phi = c,\ \ (\dfrac{\pi}{2} < c < \pi)$



$\phi = c, \ \ (0 \le c \le \pi)$

Solution.

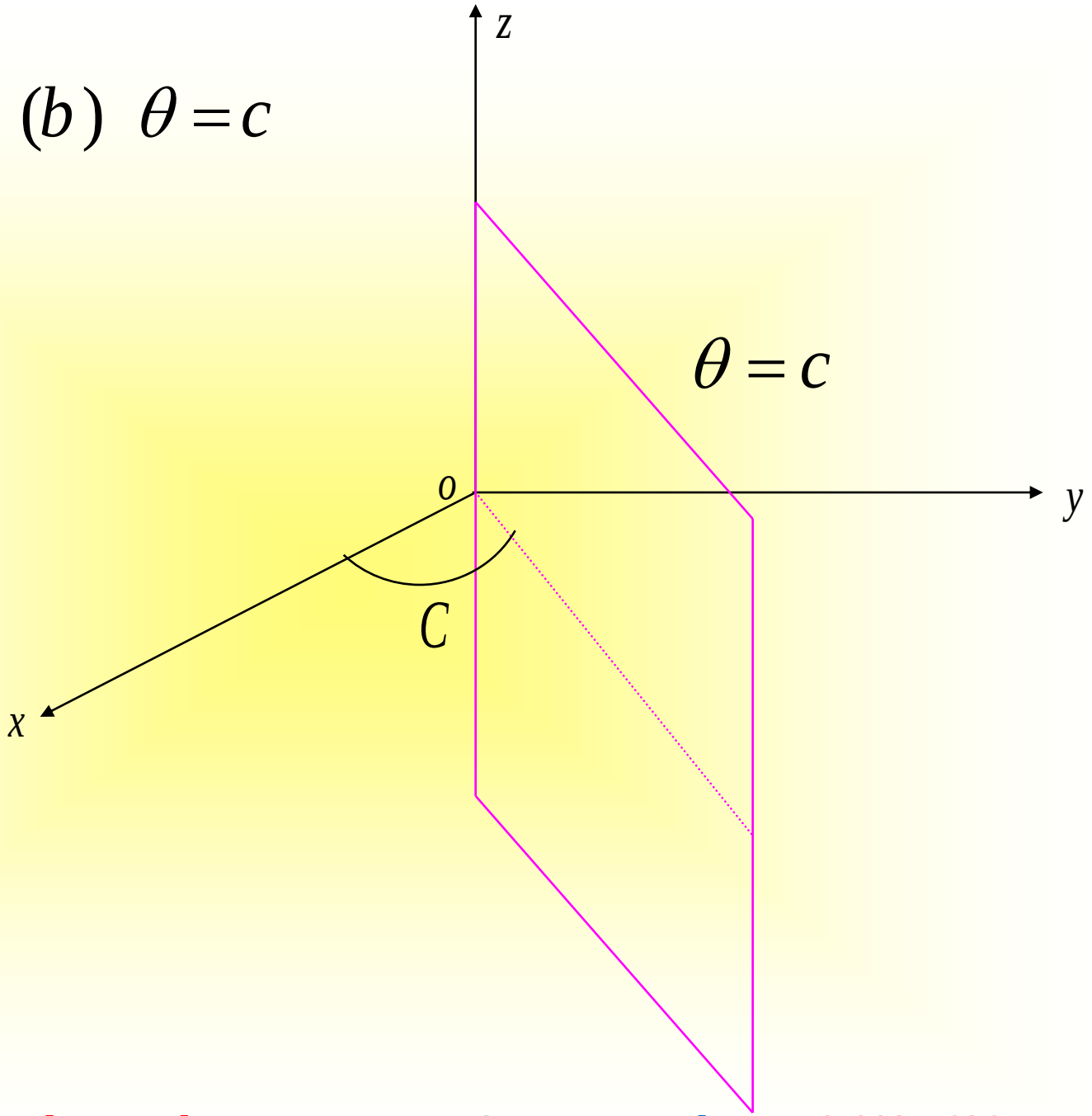
$c = \dfrac{\pi}{2}$ **xy-plane**

$c = 0$ **the positive side of z-axis**

$c = \pi$ **the negative side of z-axis**

$(b)\ \theta = c$

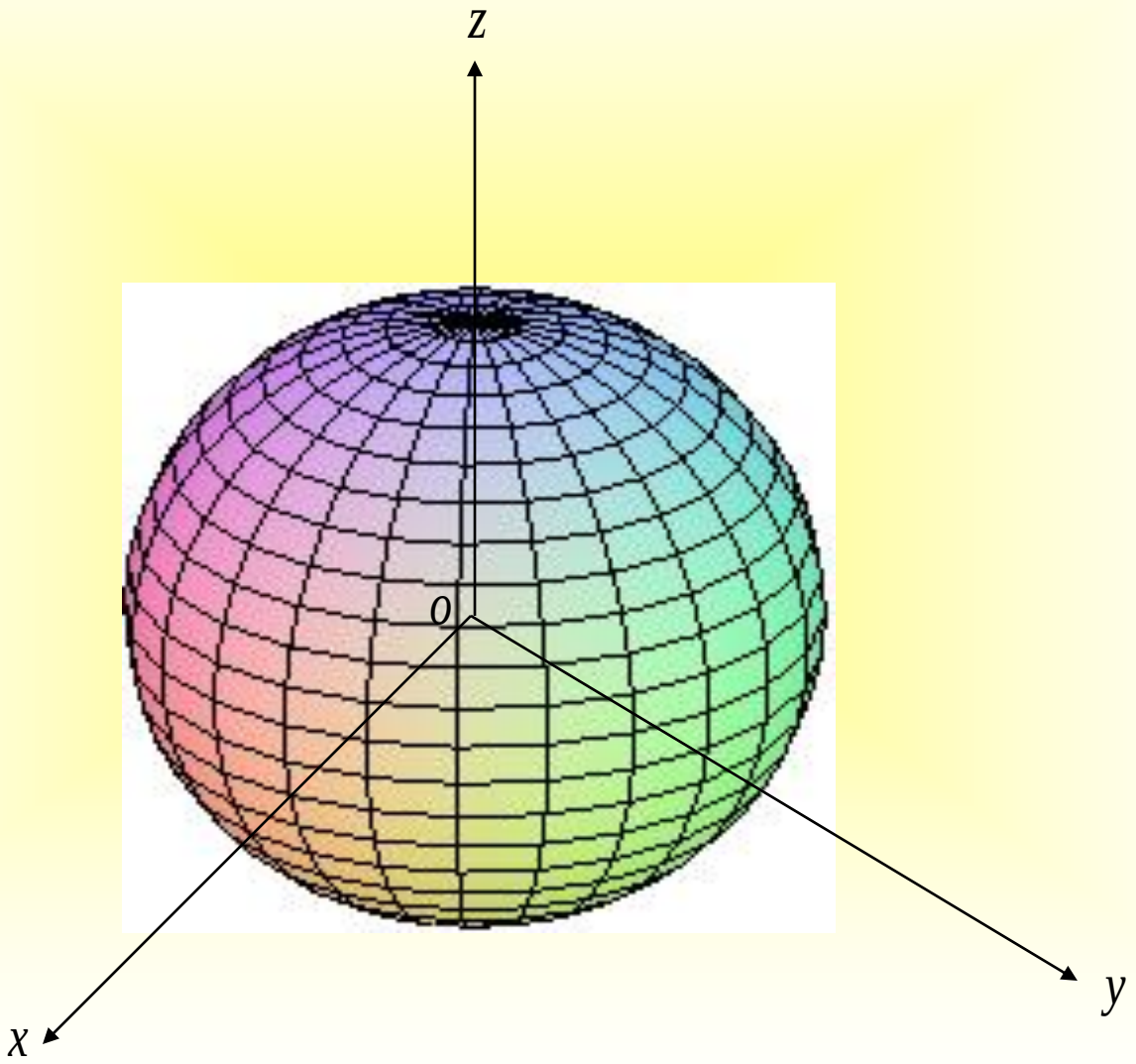
Solution.



$$(c)\ \ \rho = c,\ \ (c > 0)$$

Solution.

A sphere with the radius c and the center of the origin.

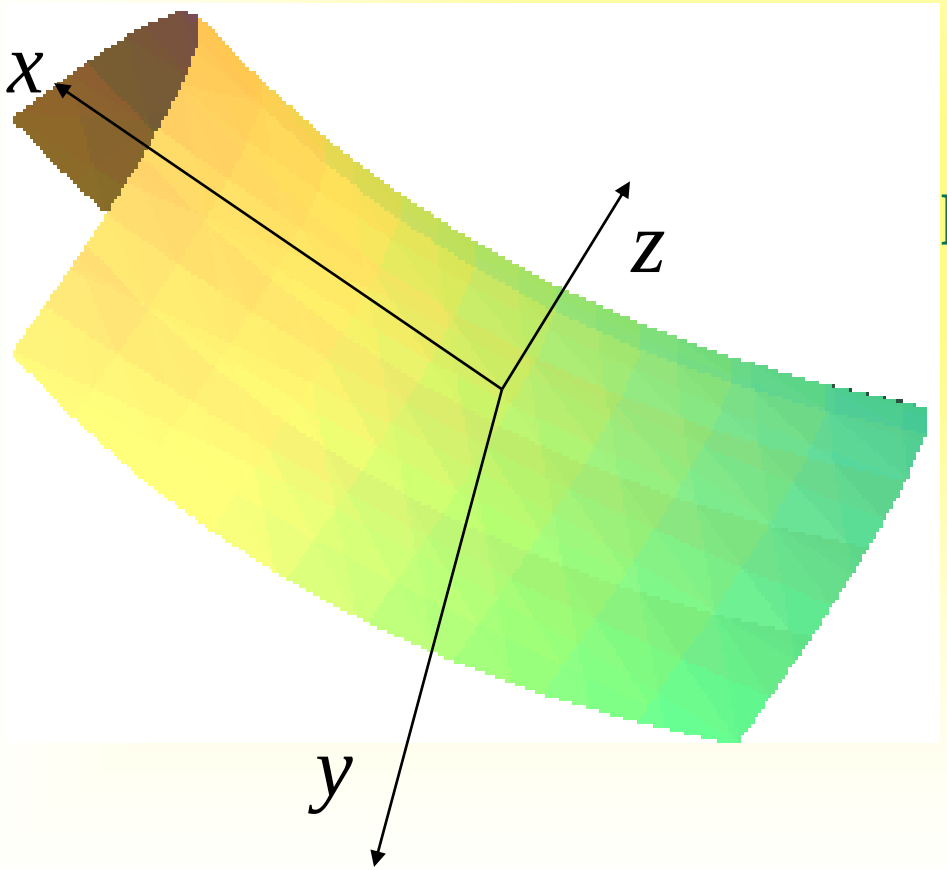


Exercise. **Draw** $\rho \sin^2 \phi . \cos 2\theta = \cos \phi$

Solution.

$$\rho \sin^2 \phi . \cos 2\theta = \cos \phi \Rightarrow \rho^2 \sin^2 \phi . (\cos^2 \theta - \sin^2 \theta) = \rho \cos \phi$$

$$\Rightarrow \rho^2 \sin^2 \phi . \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = \rho \cos \phi \Rightarrow x^2 - y^2 = z$$



perbolic paraboloid

Exercise. Change the following cartisian equations to cylindrical coordinates.

$$(a)\, x^2+y^2+z^2-8x=0 \qquad (b)\, x^2+y^2=2z$$

Solution. $(a)\ x^2+y^2+z^2-8x=0 \Rightarrow \rho^2-8\rho\sin\varphi\cos\theta=0$

$$\Rightarrow \rho^2=8\rho\sin\phi\cos\theta, \quad \vee \quad \rho=0$$

$$(b)\ x^2+y^2=2z \Rightarrow \rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)=2\rho\cos\varphi$$

$$\Rightarrow \rho=0, \ \vee \ \rho\ \sin^2\phi=2\cos\phi$$

Exercise. **Find the Cartesian and cylindrical coordinates of the spherical point** $(\sqrt{3},-\frac{\pi}{4},\frac{\pi}{3})$

Solution.

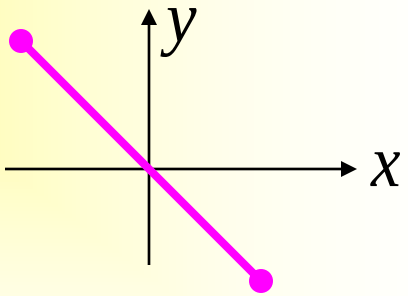
$$x=\rho\sin\phi\cos\theta=\sqrt{3}\sin\frac{\pi}{3}\cos(-\frac{\pi}{4})=\frac{3\sqrt{2}}{4}$$

$$y=\rho\sin\phi\sin\theta==\sqrt{3}\sin\frac{\pi}{3}\sin(-\frac{\pi}{4})=-\frac{3\sqrt{2}}{4}$$

$$z=\rho\cos\phi=\sqrt{3}\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

$$(\frac{3\sqrt{2}}{4},-\frac{3\sqrt{2}}{4},\frac{\sqrt{3}}{2})$$ **Cartesian**

$$r=\rho\sin\varphi=\sqrt{3}\sin\frac{\pi}{3}=\frac{3}{2}$$



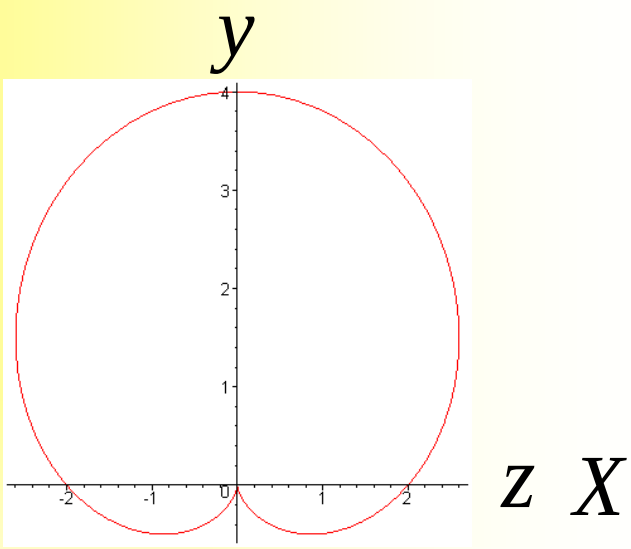
$$(\frac{3}{2},-\frac{\pi}{4},\frac{\sqrt{3}}{2})$$ **Cylindrical**

Exercise. **Draw the spherical surface** $\rho = 2 + 2\sin\phi$

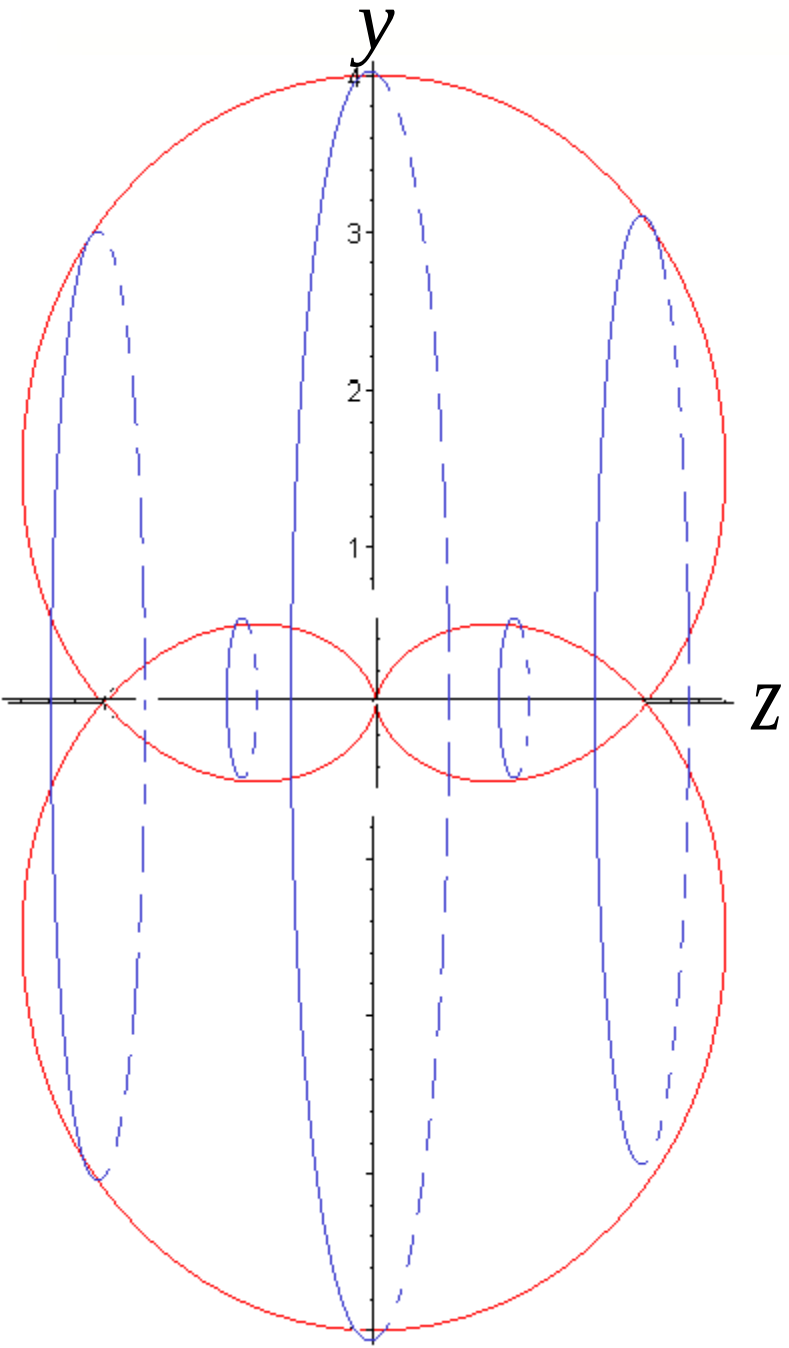
Solution.

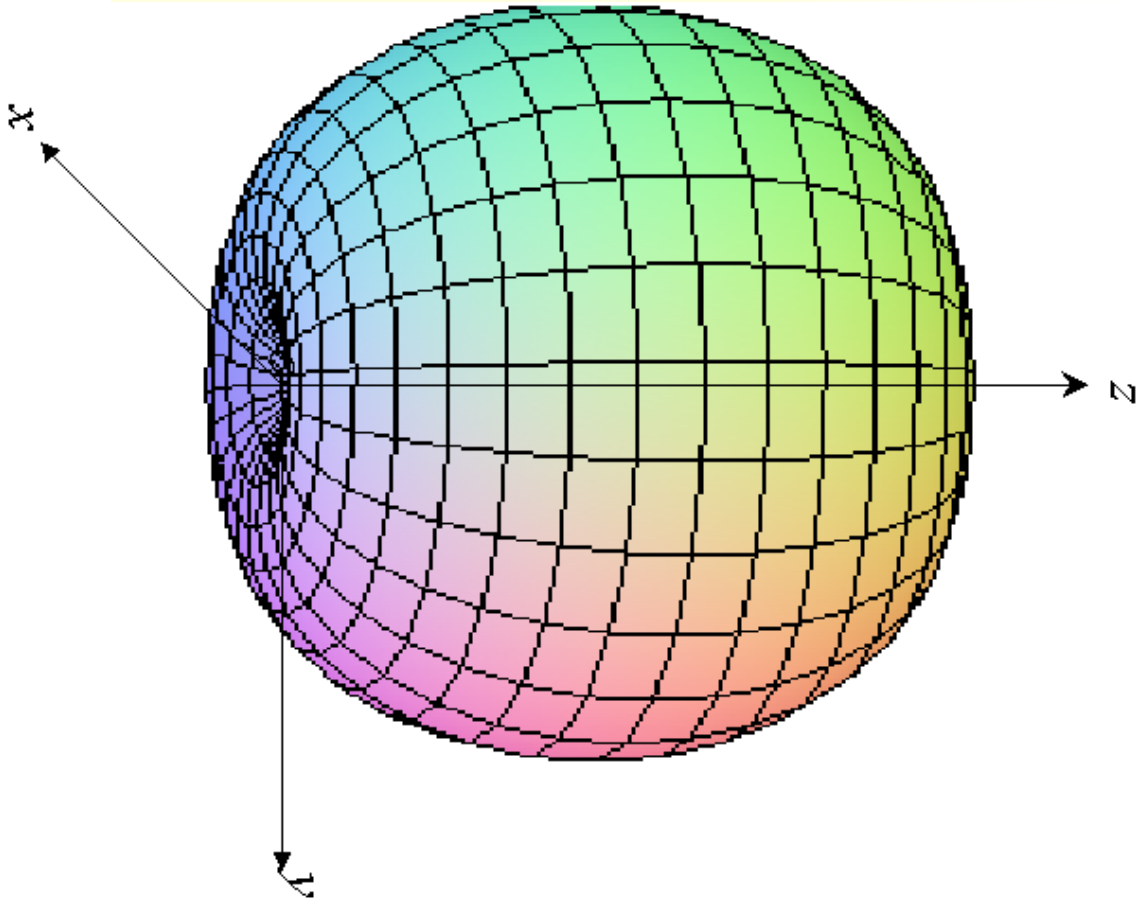
First suppose $\theta = \frac{\pi}{2}$ and in the zy $-$plane, draw the curve

$$r = 2 + 2\sin\theta$$



Now rotate this curve about z-axis.


y
4
3
2
1
z


x
z
y