Derivative: Definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If a particular point a is given, then use the following formula:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Exercise. Use the definition to find the derivative of $f(x) = x^2 + 3x + 1$

Solution. $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2+3(x+\Delta x)+1-\left(x^2+3x+1\right)}{\Delta x}$

$$=\lim_{\Delta x\to 0}\frac{x^2+2\Delta x.x+(\Delta x)^2+3x+3\Delta x)+1-x^2-3x-1}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x(2x+(\Delta x)+3)}{\Delta x}=\lim_{\Delta x\to 0}(2x+(\Delta x)+3)=2x+3$$

EXAMPLE: Given $f(x) = 3x^2 + 12$, find $f'(2)$

Solution. $$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(3x^2 + 12) - 24}{x - 2}$$

$$= \lim_{x \to 2} \frac{3x^2 - 12}{x - 2} = 3 \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}$$

$$= 3 \lim_{x - 2} (x + 2) = 12$$

$$f_{+}'(a)=\lim_{x\to a^{+}}\frac{f(x)-f(a)}{x-a}$$ right derivative at the point $x=a$;

$$f_{-}'(a)=\lim_{x\to a^{-}}\frac{f(x)-f(a)}{x-a}$$ right derivative at the point $x=a$.

Exercise. find a and b such that the function $f(x)=\begin{cases} ax+b & x<2 \\ 2x^2-1 & x\geq 2 \end{cases}$

is differentiable at any points.

Solution. for $x \neq 2$, the fuction is a polynomial, so it is differentiable.

now suppose a=2: Since f is difentiable, it is continious, so

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2), \textit{and} \lim_{\substack{x\to 2^- \\ x<2}} f(x)=\lim_{x\to 2^-} ax+b=2a+b$$

$$\lim_{x\to 2^+} f(x)=\lim_{\substack{x\to 2^+ \\ x>2}} f(x)=\lim_{x\to 2^+}(2x^2-1)=2\times 2^2-1=7$$

$$\Rightarrow 2a+b=7 \Rightarrow b-7=-2a \qquad (1)$$

$$f_+'(2)=\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{2x^2-1-7}{x-2}=\lim_{x\to 2^+}\frac{2x^2-8}{x-2}$$

$$=\lim_{x\to 2^+}\frac{2(x-1)(x+2)}{x-2}=\lim_{x\to 2^+} 2(x+2)=8. \qquad (2)$$

$$f_-'(2)=\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{ax+b-7}{x-2}$$

, and by (1), $b-7=-2a$, so:

$$f_-'(2)=\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{ax-2a}{x-2}=\lim_{x\to 2^-}\frac{a(x-2)}{x-2}=\lim_{x\to 2^-}a=a \quad (3)$$

$$f_+'(2)=f_-'(2)=\xrightarrow{(2),(3)}a=8.$$

$$b-7=-2a\overset{a=8}{\Rightarrow}b=7-2\times 8\Rightarrow b=-9$$

Formulas:

1) $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

2) $\left(f(x)g(x)\right)' = f'(x)g(x) + g'(x)f(x)$

3) $\left(cf(x)\right)' = cf'(x)$

4) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - g'(x)f(x)}{\left(g(x)\right)^2}$

5) $f(x) = c$ constant $\Rightarrow f'(x) = 0$

6) $\left(x^{n}\right)' = nx^{n-1}$

7) $\left(|x|\right)' = \dfrac{x}{|x|} = \dfrac{|x|}{x},\ x \neq 0$

8) $\left(\sin x\right)' = \cos x$

9) $\left(\cos x\right)' = -\sin x$

10) $\left(\tan x\right)' = \sec^{2} x$

11) $\left(\cot x\right)' = -\csc^{2} x$

12) $\left(\sec x\right)' = \sec x . \tan x$

13) $\left(\csc x\right)' = -\csc x . \cot x$

EXAMPLE Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned} \frac{d}{dx}[(x^2+1)(x^3+3)] &= (x^2+1)(3x^2) + (x^3+3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x. \end{aligned}$$

EXAMPLE : Given $f(x) = \dfrac{x^2 - 27}{x - 6}$ find $f'(x)$

$$f'(x) = \frac{2x(x-6) - (x^2 - 27)}{(x-6)^2} = \frac{x^2 - 12x + 27}{(x-6)^2} = \frac{(x-3)(x-9)}{(x-6)^2}$$

Example : Differentiate $\dfrac{\sqrt{t}}{2t+3}$

Solution: $\left(\dfrac{\sqrt{t}}{2t+3}\right)' = \left(\dfrac{t^{\frac{1}{2}}}{2t+3}\right)'$

$$\frac{(2t+3)\left(t^{\frac{1}{2}}\right)' - \left(t^{\frac{1}{2}}\right)(2t+3)'}{(2t+3)^2} =$$

$$\frac{(2t+3)\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}}(2)}{(2t+3)^2} = \frac{(2t+3)\frac{1}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} =$$

$$\frac{\frac{t}{\sqrt{t}} + \frac{3}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} = \frac{\sqrt{t} + \frac{3}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} = \frac{\frac{3}{2\sqrt{t}} - \sqrt{t}}{(2t+3)^2} =$$

$$\frac{\frac{3}{2\sqrt{t}} - \sqrt{t}}{(2t+3)^2}\left(\frac{2\sqrt{t}}{2\sqrt{t}}\right) = \frac{3-2t}{2\sqrt{t}(2t+3)^2}$$

Exercise. $f(x)=\sqrt[3]{(x+1)^2}\,(x-2)^{\frac{1}{3}},\quad f'(x)=?$

Solution. $\Rightarrow f(x)=(x+1)^{\frac{2}{3}}(x-2)^{\frac{1}{3}}\Rightarrow$

$$f'(x)=\frac{2}{3}(x+1)^{\frac{2}{3}-1}(x-2)^{\frac{1}{3}}+\frac{1}{3}(x-2)^{\frac{1}{3}-1}(x+1)^{\frac{2}{3}}$$

Exercise.

$$f(x)=\begin{cases} x+\sin x & x\geq 0 \\ \tan x & x<0 \end{cases}\qquad f'(x)=?$$

Solution.

$$f(x)=\begin{cases}1+\cos x & x>0\\ \sec^2 x & x<0\end{cases}.\quad \textit{For } x=2, \text{we must use the definition:}$$

$$f_+'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x+\sin x-(0+\sin 0)}{x-0}=\lim_{x\to 0^+}\frac{x}{x}+\frac{\sin x}{x}=2;$$

$$f_+'(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{\tan x-\sin 0}{x-0}=\lim_{x\to 0^-}\frac{\tan x}{x}=$$

$$\lim_{x\to 0^-}\frac{\frac{\sin x}{\cos x}}{x}=\lim_{x\to 0^-}\frac{1}{\cos x}\frac{\sin x}{x}=1\times 1=1;$$

$$\Rightarrow f_+'(0)=2\neq 1=f_1'(0)\Rightarrow f'(0)$$ deos not exist!

Exercise. $y = \dfrac{\cos x}{1 - \sin x}$ $\qquad \dfrac{dy}{dx} = ?$

Solution.

$$\frac{dy}{dx} = \frac{(1 - \sin x)\dfrac{d}{dx}(\cos x) - \cos x\dfrac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$

$$= \frac{1}{1 - \sin x}$$