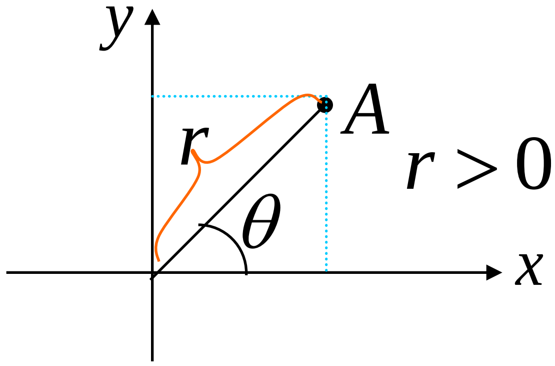
Polar Coordinates

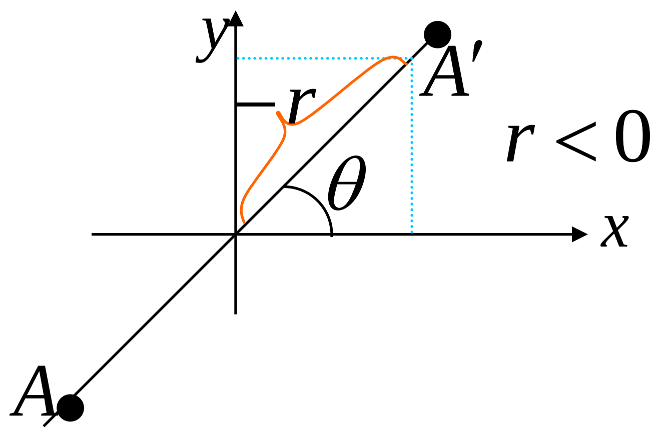
Each point is shown by $A(r,\theta)$ **where** θ **is the angle between oA and the positive side of the x-axis in radians and** $\theta \in (-\infty,+\infty)$ **and either**

(a) r>0 , where in this case $r = |OA| > 0$



or

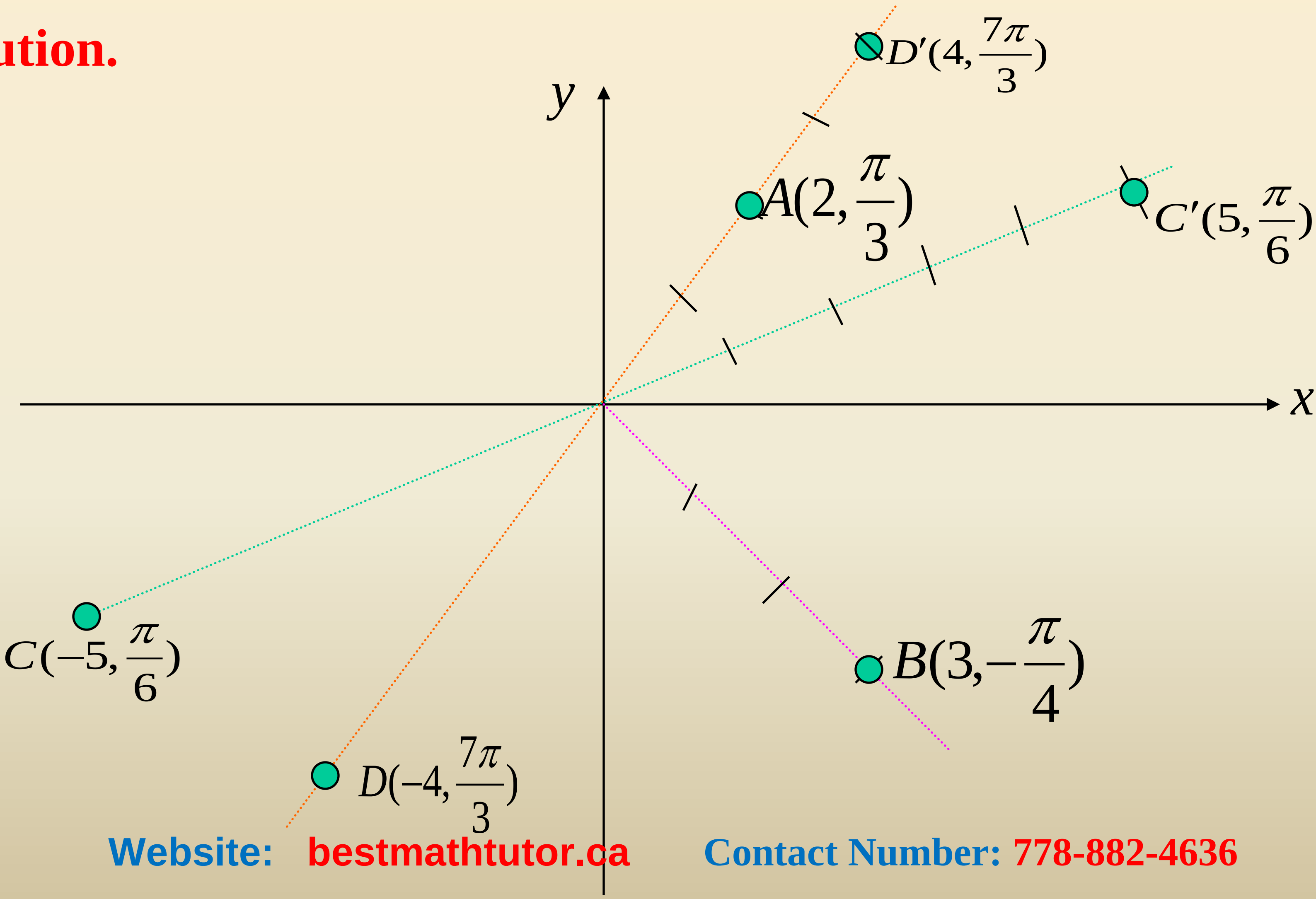
(b) r<0, and in this case we find the point $A'(-r,\theta)$ **from part (a), and then we reflect** A' **in the origin to get the point** $A(r,\theta)$



Exercise. **Plot the points having the following polar coordinates**

$$A(2, \frac{\pi}{3}) \quad B(3, -\frac{\pi}{4}) \quad C(-5, \frac{\pi}{6}) \quad D(-4, \frac{7\pi}{3})$$

Solution.



Exercise. **Plot the following polar coordinates equations:**

(a) $r = C$ constant

$(b)\ \theta = C$ constant

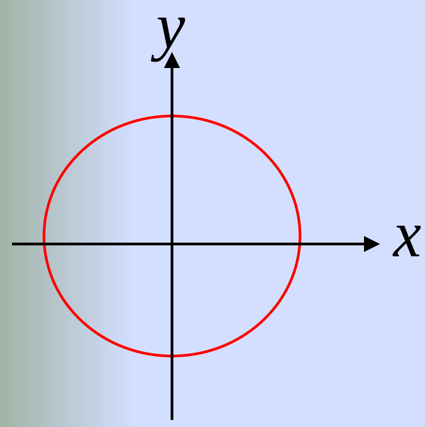
$(c)\ 0 \le C_1 < r \le C_2,\quad C_1, C_2$ constant

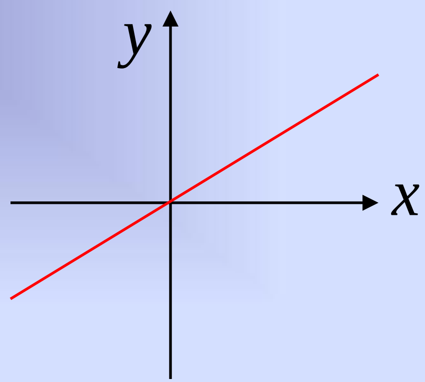
$(d)\ C_1 \le \theta < C_2\quad C_1, C_2$ constant

Solution.

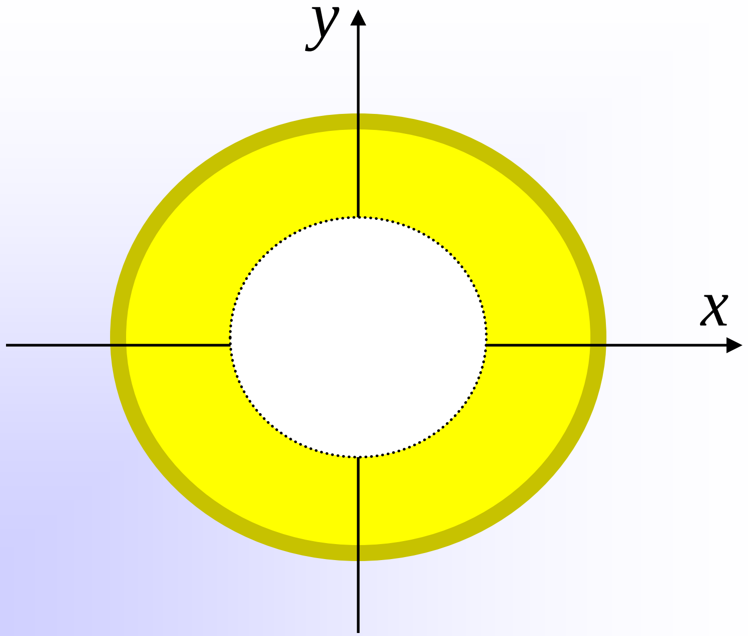
(a) $r = C$ constant

$(b)\ \theta = C$ constant

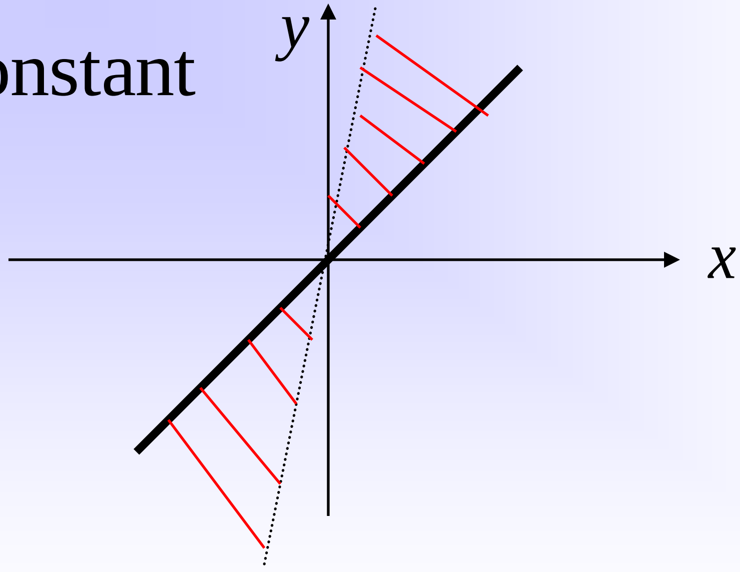




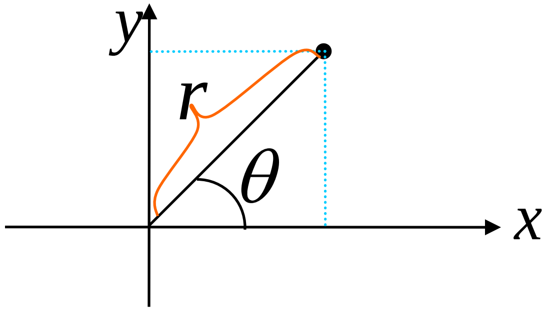
$(c)\ 0 \le C_1 < r \le C_2, \quad C_1, C_2$ constant



$(d)\ C_1 \le \theta < C_2 \quad C_1, C_2$ constant



Converting Formulas:



$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x}$$

Exercise. **Change the following polar coordinates to Cartesian coordinates and then graph :**

$(a)\ r = a\sin\theta$

$(b)\ r = a\cos\theta$

$(c)\ r\cos(\frac{\pi}{4} + \theta) = \sqrt{2}$

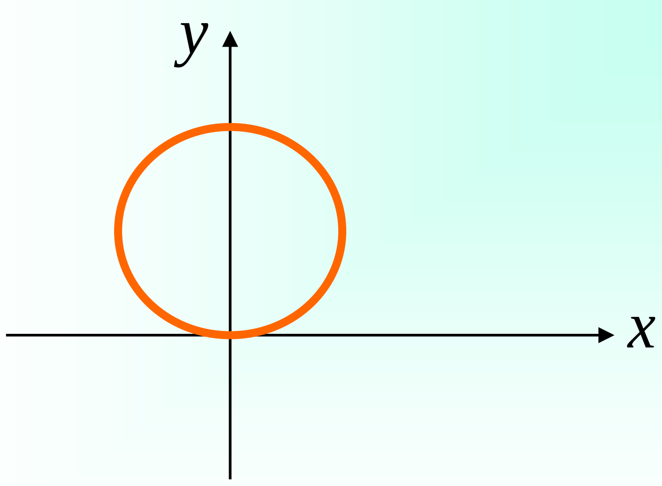
Solution. (a) $r = a\sin\theta$

$$r = a\sin\theta \Rightarrow r^2 = ar\sin\theta \Rightarrow x^2 + y^2 = ay \Rightarrow x^2 + y^2 - ay = 0$$

$$\Rightarrow x^2 + (y - \frac{a}{2})^2 - \frac{a^2}{4} = 0 \Rightarrow x^2 + (y - \frac{a}{2})^2 = \frac{a^2}{4}$$

A circle with the radius of

$(0, \frac{a}{2})$ and the radius of $\left|\frac{a}{2}\right|$



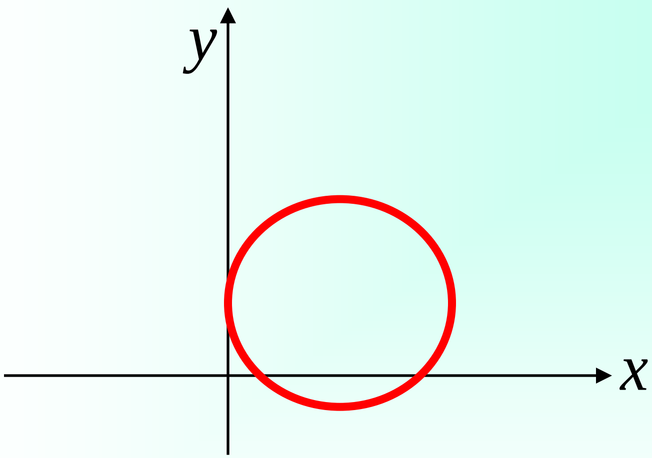
$(b)\ \ r = a\cos\theta$

$$r = a\cos\theta \Rightarrow r^2 = ar\cos\theta \Rightarrow x^2 + y^2 = ax \Rightarrow x^2 - ax + y^2 = 0$$

$$\Rightarrow (x - \frac{a}{2})^2 - \frac{a^2}{4} + y^2 = 0 \Rightarrow (x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$$

A circle with the radius of

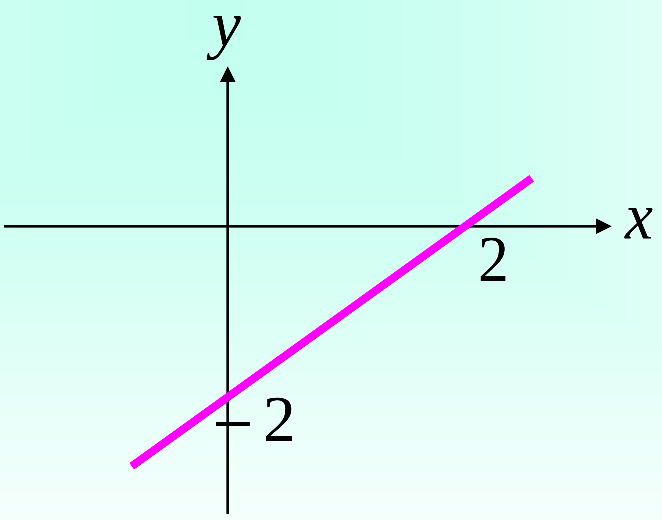
$(\frac{a}{2}, 0)$ and the radius of $\left|\frac{a}{2}\right|$



$$(c)\ \ r\cos(\frac{\pi}{4}+\theta)=\sqrt{2}\Rightarrow r(\cos\frac{\pi}{4}.\cos\theta-\sin\frac{\pi}{4}.\sin\theta)=\sqrt{2}$$

$$\Rightarrow r(\frac{\sqrt{2}}{2}.\cos\theta-\frac{\sqrt{2}}{2}.\sin\theta)=\sqrt{2}\Rightarrow r\cos\theta-r\sin\theta=2$$

$\Rightarrow x-y=2$ **A straight Line**



Exercise. **Change the following Cartesian coordinates to polar coordinates :**

$(a)\ x^3y^2 - x^2 + y^2 = 5$ $(b)\ 2x^2 + 2y^2 - x + y^3 = 0$

Solution.

$(a)\ x^3y^2 - x^2 + y^2 = 5 \Rightarrow$

$(r\cos\theta)^3(r\sin\theta)^2 - (r\cos\theta)^2 + (r\sin\theta)^2 = 5$

....

$(b)\ 2x^2 + 2y^2 - x + y^3 = 0 \Rightarrow\ 2r^2 - (r\cos\theta) + (r\sin\theta)^3 = 0$

$\Rightarrow 2r - \cos\theta + r^2\sin^3\theta = 0$ or $r = 0$

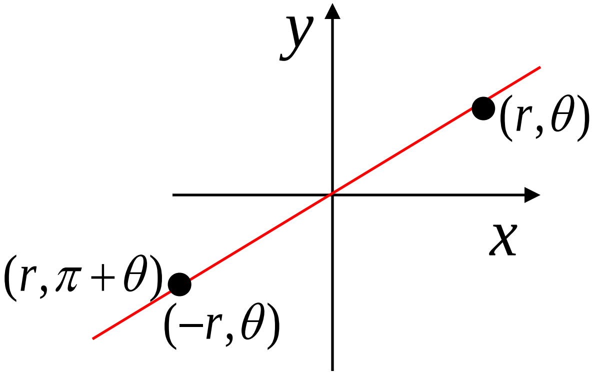
Graphing the Polar Equation $f(r,\theta)=0$

To graph the equation follow the following 6 steps:

(1) If $f(r,\theta)=f(r,\pi+\theta+2k_0\pi)$ **or** $f(r,\theta)=f(-r,\theta+2k_0\pi)$

For some $k_0 \in Z$

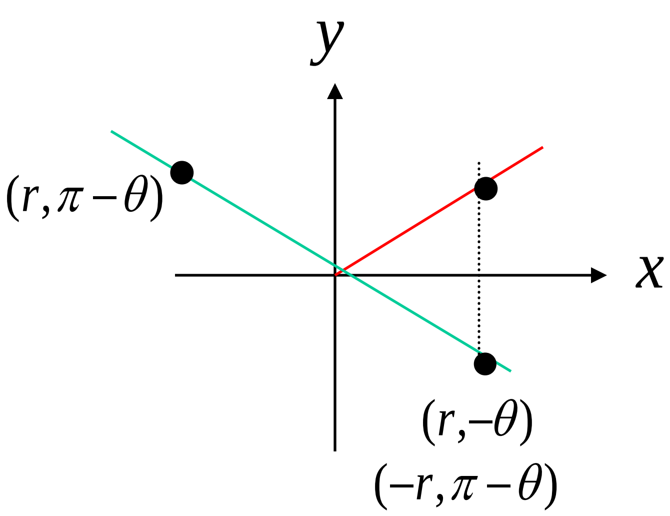
then the graph is symmetric with respect to the origin (pole)



(2) If $f(r,\theta) = f(-r, \pi - \theta + 2k_0\pi)$ **or** $f(r,\theta) = f(r, -\theta + 2k_0\pi)$

For some $k_0 \in Z$

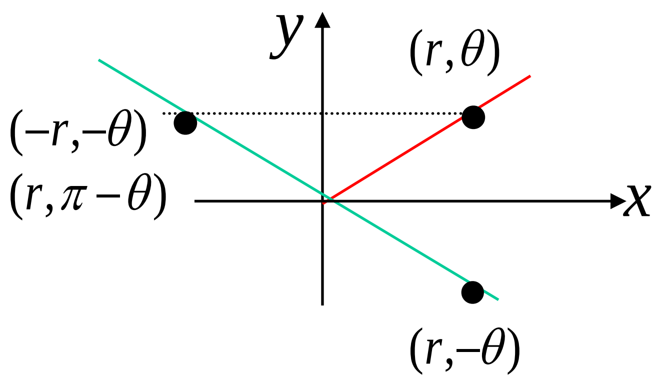
then the graph is symmetric with respect to the x-axis (polar axis)



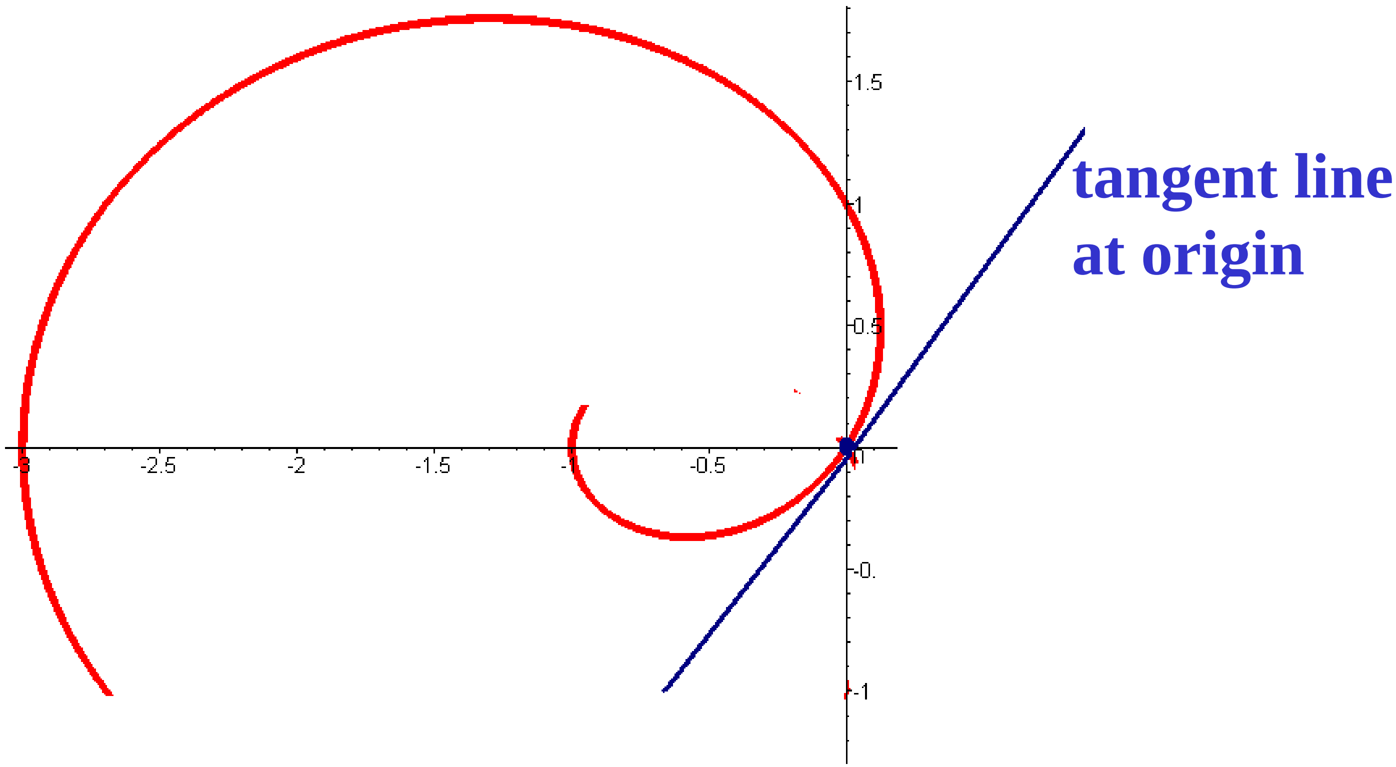
(3) If $f(r,\theta) = f(r, \pi - \theta + 2k_0\pi)$ **or** $f(r,\theta) = f(-r, -\theta + 2k_0\pi)$

For some $k_0 \in Z$

then the graph is symmetric with respect to the y-axis ($\frac{\pi}{2}$ axis)



(4) Put r=0 and find θ and if $\theta = \theta_1$ is an answer, this means that the graph is tangent to the line $\theta = \theta_1$ at origin



(5) Use the equation and find the maximum and minimum of r if possible.

(6) Draw the following table values

θ	
r	

and find the values of r for some arbitrary values of θ

Exercise. Graph the polar equation $r = 1 - 2\cos\theta$.

Solution.

$$\left.\begin{array}{l} -r = 1 - 2\cos(\theta + 2k_0\pi) \Rightarrow r = -1 + 2\cos\theta \\ r = 1 - 2\cos(\theta + \pi + 2k_0\pi) \Rightarrow r = 1 + 2\cos\theta \end{array}\right\} \neq r = 1 - 2\cos\theta$$

So the graph is not symmetric with respect to the origin (pole)

$$r = 1 - 2\cos(-\theta) \Rightarrow r = 1 - 2\cos\theta \Rightarrow$$

So the graph is symmetric with respect to the x-axis (polar axis)

$$\left.\begin{array}{l} r = 1 - 2\cos(\pi - \theta + 2k_0\pi) \Rightarrow r = 1 + 2\cos\theta \\ -r = 1 - 2\cos(-\theta + 2k_0\pi) \Rightarrow r = -1 + 2\cos\theta \end{array}\right\} \neq r = 1 - 2\cos\theta \Rightarrow$$

So the graph is not symmetric with respect to the y-axis ($\frac{\pi}{2}$ axis)

$$r=0\Rightarrow 1-2\cos\theta=0\Rightarrow \cos\theta=\frac{1}{2}\Rightarrow \theta=\frac{\pi}{3}\quad \text{Answer in } (0,\pi)$$

Tangent line at origin

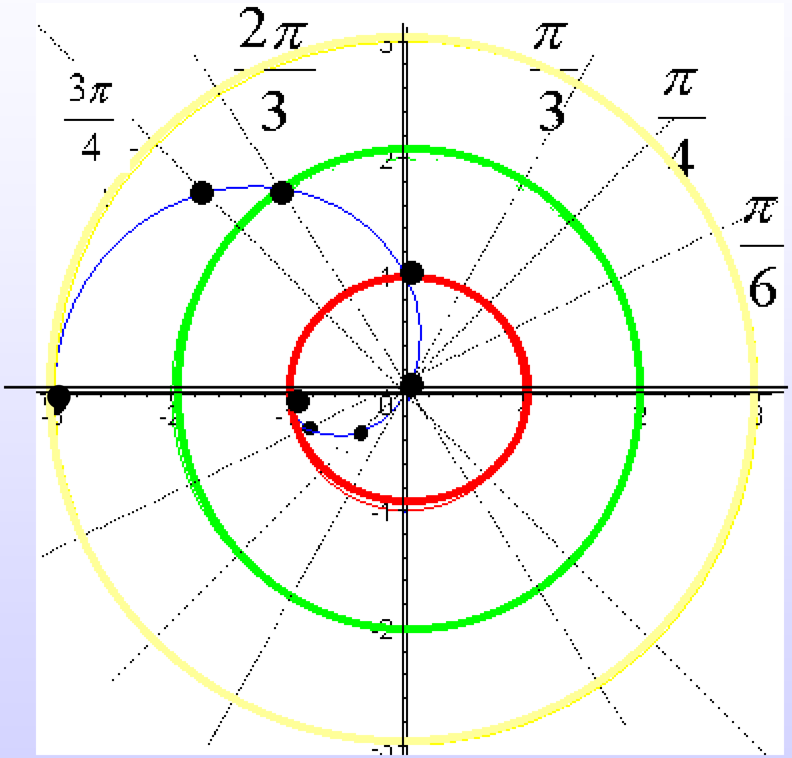
$$r=1-2\cos\theta,\quad \left.\begin{array}{l}\cos\theta=1\Rightarrow r=1-2=-1\\ \cos\theta=-1\Rightarrow r=1+2=3\end{array}\right\}\Rightarrow$$

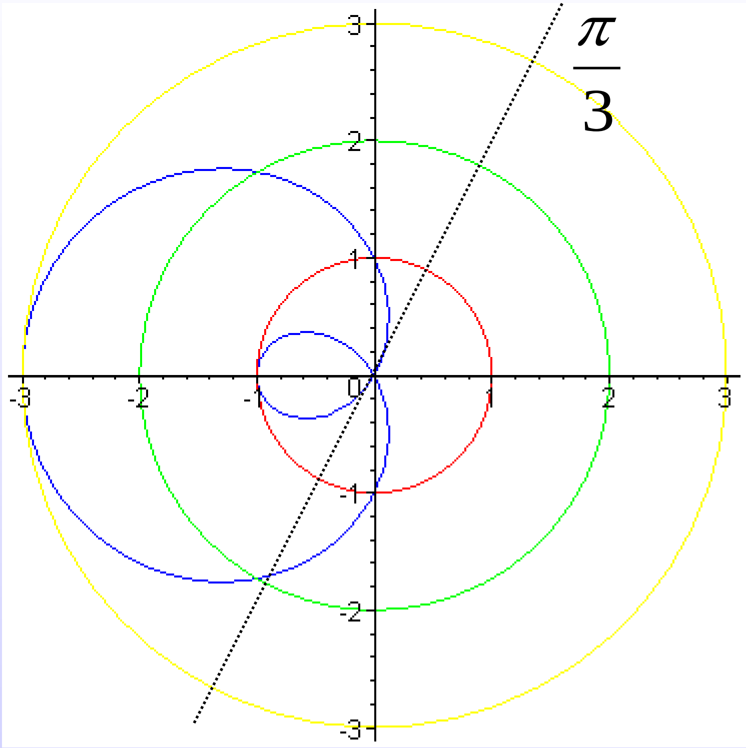
$$Min\ r=-1 \text{ and } Max\ r=3.$$

Table Values: $r=1-2\cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
r	-1	$1-\sqrt{3}$	$1-\sqrt{2}$	0	1	2	$1+\sqrt{2}$	3

Graph of the polar equation $r = 1 - 2\cos\theta$.





The above graph is called Limacon

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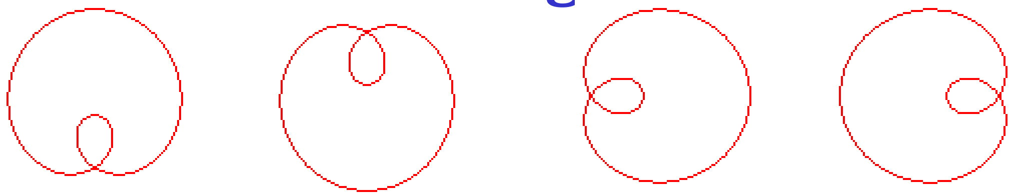
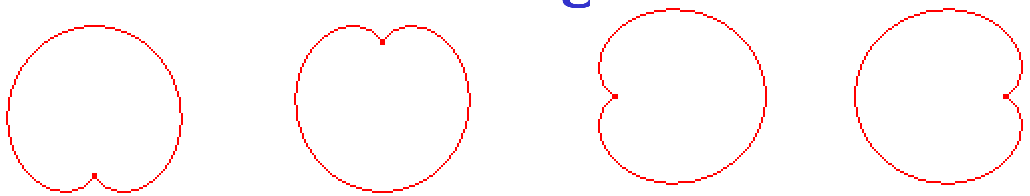
In general $r = a \pm b\sin\theta, \quad$ and $\; r = a \pm b\cos\theta, \quad (a, b > 0)$

is a Limacon and we have:

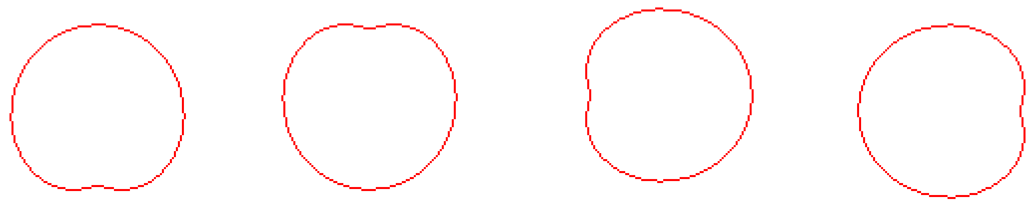
1) If $0 < \dfrac{a}{b} < 1$ **, then the Limacon has a loop and the graph is one of the following:**



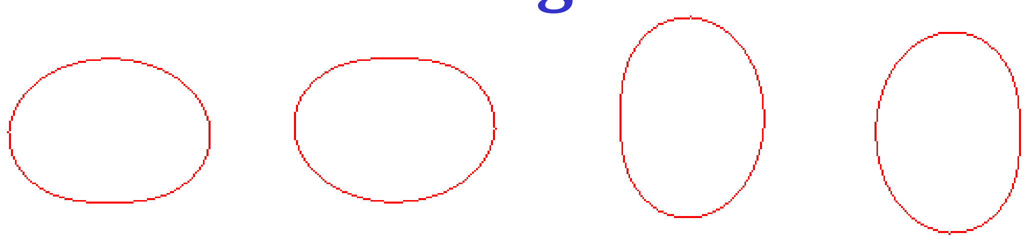
2) If $\dfrac{a}{b} = 1$**, then the Limacon is a Cardioid and the graph is one of the following:**



3) If $1 < \frac{a}{b} < 2$ **, then the Limacon has a dent and the graph is one of the following:**



4) If $2 \leq \frac{a}{b}$ **, then the Limacon is convex and the graph is one of the following:**



Exercise. Graph the polar equation $r = 3 + 2\sin\theta$.

Solution.

$a = 3,\ \ b = 2 \Rightarrow 1 < \dfrac{a}{b} = \dfrac{3}{2} < 2,$ **so the Limacon has a dent.**

$$\left.\begin{array}{l} -r = 3 + 2\sin(\theta + 2k_0\pi) \Rightarrow r = -3 - 2\sin\theta \\ r = 3 + 2\sin(\pi + \theta + 2k_0\pi) \Rightarrow r = 3 - 2\sin\theta \end{array}\right\} \Rightarrow$$

So the graph is not symmetric with respect to the origin (pole)

$$\left.\begin{array}{l} r = 3 + 2\sin(-\theta + 2k_0\pi) \Rightarrow r = 3 - 2\sin\theta \\ -r = 3 + 2\sin(\pi - \theta + 2k_0\pi) \Rightarrow -r = 3 + 2\sin\theta \end{array}\right\} \Rightarrow$$

So the graph is not symmetric with respect to the x-axis (polar axis)

$$r = 3 + 2\sin(\pi - \theta) \Rightarrow r = 3 + 2\sin\theta$$

So the graph is symmetric with respect to the y-axis ($\frac{\pi}{2}$ axis)

$$r = 3 + 2\sin\theta, \quad \left.\begin{aligned} \sin\theta = 1 \Rightarrow r = 3 + 2 = 5 \\ \cos\theta = -1 \Rightarrow r = 3 - 2 = 1 \end{aligned}\right\} \Rightarrow$$

$$Max \ \ r = 5, \qquad Min \ \ r = 1$$

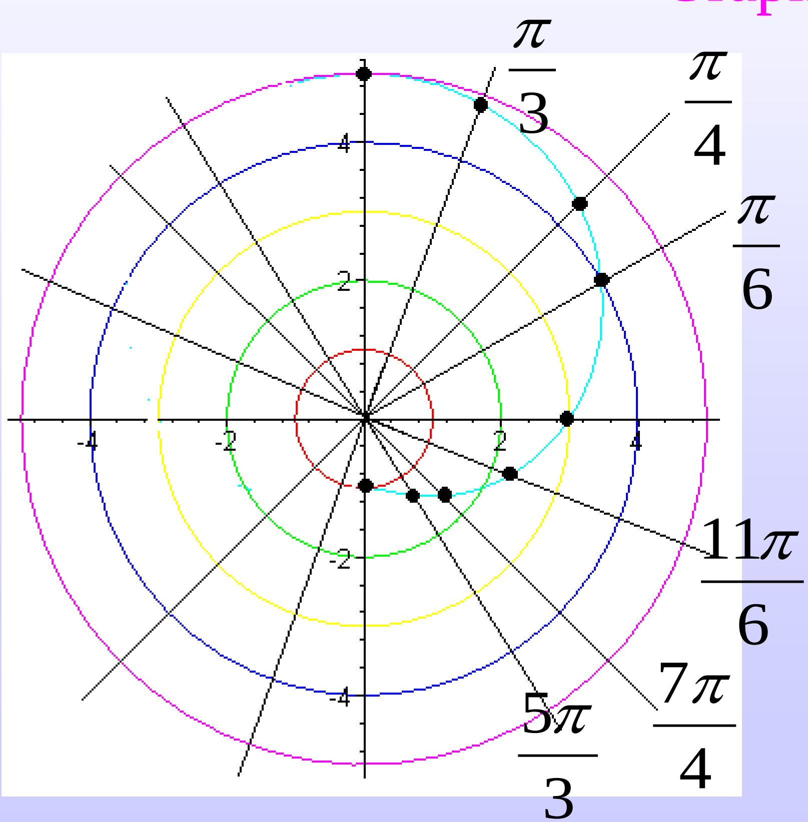
$$r = 0 \Rightarrow 3 + 2\sin\theta = 0 \Rightarrow \sin\theta = \frac{3}{2} \text{ impossible}$$

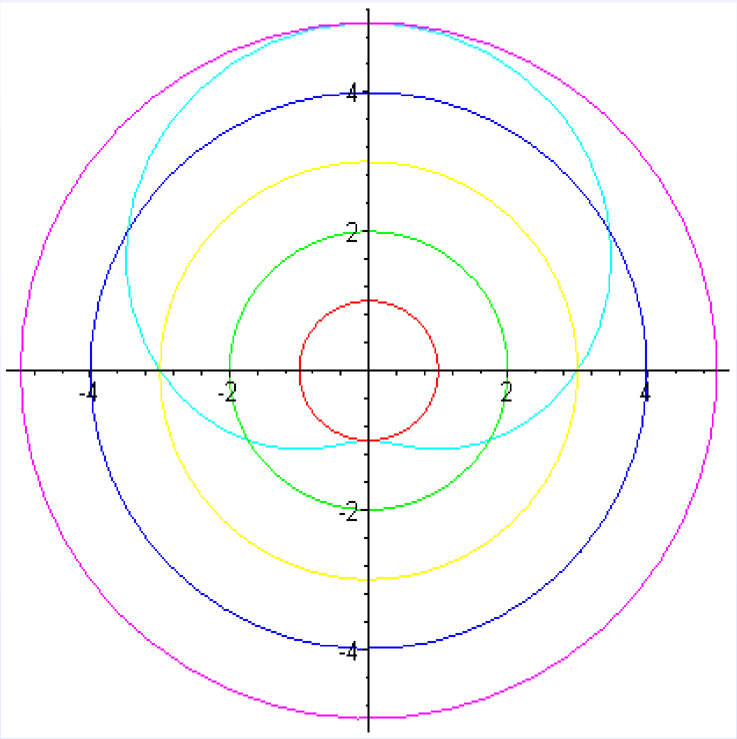
So there is no tangent line at origin

Table Values of $r = 3 + 2\sin\theta$

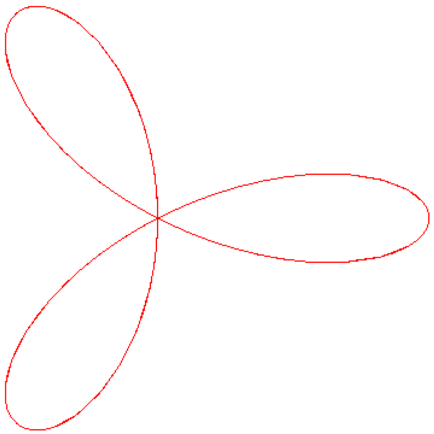
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	3	4	$3+\sqrt{2}$	$3+\sqrt{3}$	5	1	$3-\sqrt{3}$	$3-\sqrt{2}$	2	3

Graph of $r = 3 + 2\sin\theta$

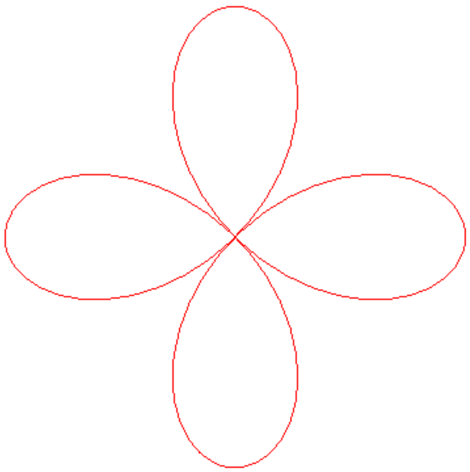




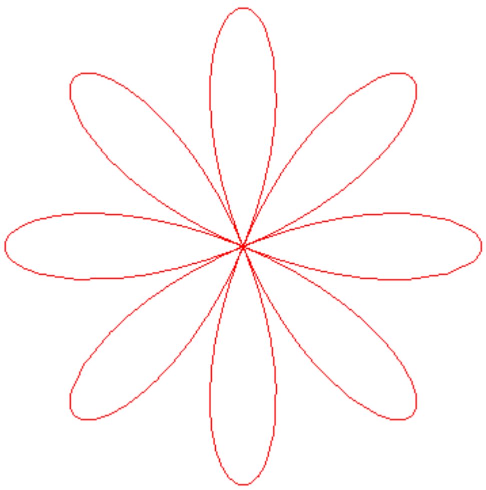
The graph of $r = a\sin n\theta \quad r = a\cos n\theta \quad a \in R, \quad n \in N, n > 1$ **is a rose with n leaves if n is odd and having 2n leaves if n is even.**



$n = 3$



$n = 2$



$n = 4$

Exercise. Graph the polar equation $r = 3\cos 2\theta$.

Solution. **n=2 so the graph is a Rose having 4 leaves.**

$$r = 3\cos 2(\pi + \theta) \Rightarrow r = 3\cos 2\theta$$

So the graph is symmetric with respect to the origin (pole)

$$r = 3\cos 2(-\theta) \Rightarrow r = 3\cos 2\theta$$

So the graph is symmetric with respect to the x-axis (polar axis)

$$r = 3\cos 2(\pi - \theta) \Rightarrow r = 3\cos 2\theta$$

So the graph is symmetric with respect to the y-axis ($\frac{\pi}{2}$ axis)

$$r = 0 \Rightarrow 3\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$ Answer in $(0, \frac{\pi}{2})$

Tangent line at origin

$Max \quad r = 3, \qquad Min \quad r = -3$

$r = 3\cos 2\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	3	$\frac{3}{2}$	0	$\frac{-3}{2}$	-3