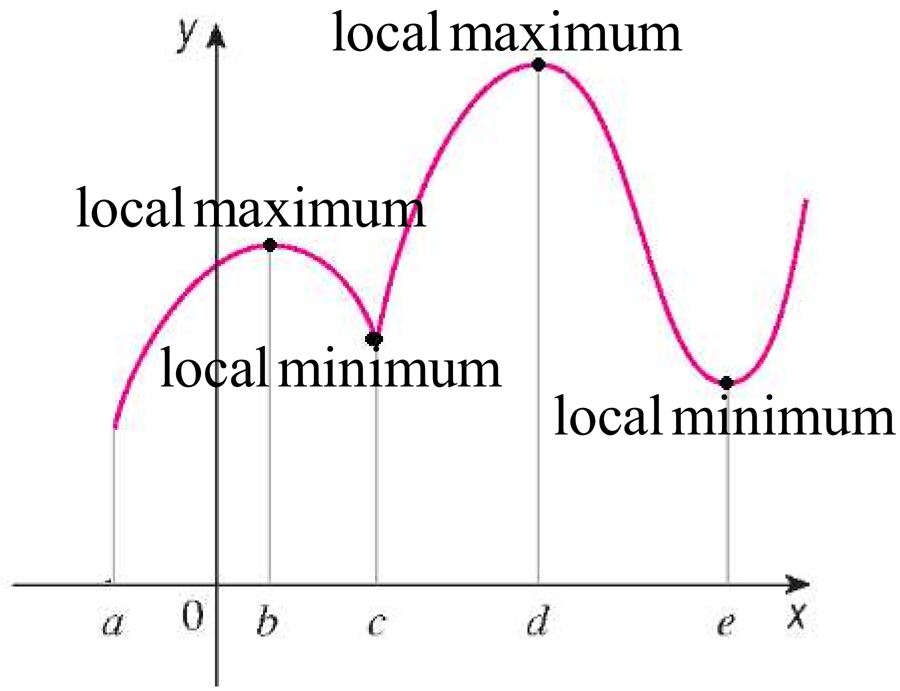
Definition The number $f(c)$ is a

local maximum value of f if $f(c) \geq f(x)$ when x is near c.

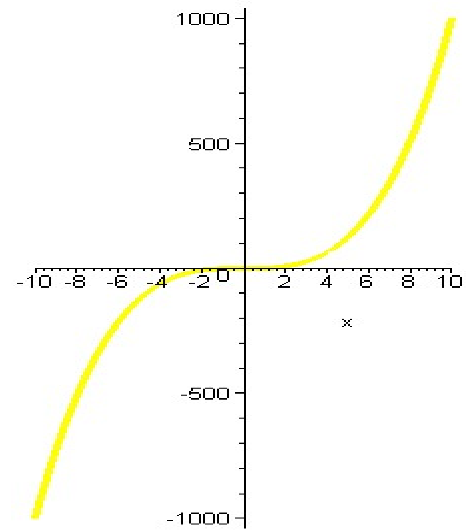
local minimum value of f if $f(c) \leq f(x)$ when x is near c.



Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

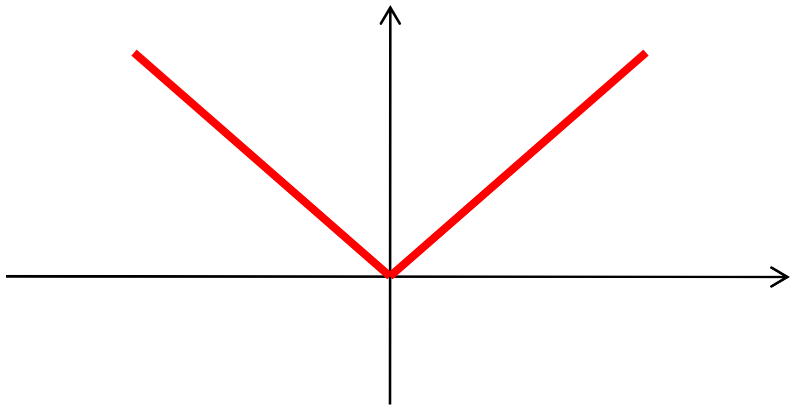
Example.

For the function $f(x)=x^3$, the point $c=0$ is not a local maximum or a local minimum, but $f'(0)=0$.



Example.

$f(x) = |x|$ has a local minimum at c=0, but f′(0) deos not exist!



Critical Point: is a point c of the domain f , where $f'(c) = 0$ or $f'(c)$ does not exist.

Exercise. $f(x)=\dfrac{x}{3+x^2}$ find the critical points

Solution.

$$f'(x)=\frac{1\times(3+x^2)-2x^2}{(3+x^2)^2}=\frac{3-x^2}{(3+x^2)^2}$$

$$f'(x)=0\Rightarrow 3-x^2=0\Rightarrow x^2=3\Rightarrow x=-\sqrt{3}\ \ or\ \ x=\sqrt{3}$$

Critical points

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Exercise.

Find the critical numbers, x-coordinates of local maxima and minima, intervals where $f(x)$ is increasing or decreasing;

$$(a)\quad f(x) = x^{\frac{2}{3}} - x^{\frac{1}{3}} \qquad (b)\quad f(x) = \begin{cases} 2x+9 & x \le -2 \\ x^2+1 & x > -2 \end{cases}$$

Solution.

$$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} - \frac{1}{3}x^{\frac{1}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{2}{3}} = \frac{2}{3x^{\frac{1}{3}}} - \frac{1}{3x^{\frac{2}{3}}} = \frac{2x-1}{3x^{\frac{2}{3}}}$$

$$f'(x)=0 \Rightarrow 2x-1=0 \Rightarrow x=\frac{1}{2}$$

$$3x^{\frac{2}{3}}=0 \Rightarrow x^{\frac{2}{3}}=0 \Rightarrow x=0.$$

Critical Points: $x=\frac{1}{2}, \quad x=0.$

x	$-\infty$	0	$\frac{1}{2}$	$+\infty$
$2x-1$	$-$	$-$	$+$	
$3x^{\frac{2}{3}}$	$+$	$+$	$+$	
$f'(x)=\dfrac{2x-1}{3x^{\frac{2}{3}}}$	$-$	$-$	$+$	
f	$\searrow$ 0	$\searrow$ local minimum	$\nearrow$	

$$(b)\quad f(x)=\begin{cases}2x+9 & x\le -2\\ x^2+1 & x>-2\end{cases}\Rightarrow f'(x)=\begin{cases}2 & x<-2\\ 2x & x>-2\end{cases}$$

$$f'(x)=0\Rightarrow 2x=0,\ \ x>-2\Rightarrow x=0,\ \ 0>-2$$

$$f_+'(-2)=\lim_{\substack{x\to -2^+\\ x>-2}}\frac{f(x)-f(-2)}{x-(-2)}=\lim_{x\to -2}\frac{2x+9-5}{x+2}=\lim_{x\to -2}\frac{2(x+2)}{x+2}=2$$

$$f_-'(-2)=\lim_{\substack{x\to -2^-\\ x<-2}}\frac{f(x)-f(-2)}{x-(-2)}=\lim_{x\to -2}\frac{x^2+1-5}{x+2}=\lim_{x\to -2}\frac{(x+2)(x-2)}{x+2}=\lim_{x\to -2}(x-2)=-4$$

$$\Rightarrow f_+'(-2)=2\ne -4=f_-'(-2)\Rightarrow f'(-2)\text{ does not exist!}$$

$$\Rightarrow \text{Critical Points: }\ x=-2,\ \ x=0.$$

x	$-\infty$		-2		0		$+\infty$
$2x$		$-$		$-$		$+$	
$f'(x)$		$+$		$-$		$+$	
f		↗	5 local maximum	↘	1 local minimum	↗	

Second Derivative Test. Suppose that $f'(c) = 0$.

$f''(c) < 0 \Rightarrow f$ has local maximum at x=c;

$f''(c) > 0 \Rightarrow f$ has local minimum at x=c.

Exercise. Find the local maximum and the local minimum of the funtion $y = \sin x + \cos x$.

Solution.

$f'(x) = \cos x - \sin x$, $f'(x) = 0 \Rightarrow \cos x = \sin x$,

divide by cosx:

$$\Rightarrow 1 = \frac{\sin x}{\cos x} \Rightarrow 1 = \tan x \Rightarrow x = k\pi + \frac{\pi}{4}, \quad k \in \mathbb{Z}.$$

$f''(x) = -\sin x - \cos x$

$$\Rightarrow f''(k\pi+\frac{\pi}{4})=-\sin(k\pi+\frac{\pi}{4})-\cos(k\pi+\frac{\pi}{4})$$

$$=\begin{cases}-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}=-\sqrt{2}<0 & \text{if } \quad k \text{ even}\\ -\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)=\sqrt{2}>0 & \text{if } \quad k \text{ odd}\end{cases}$$

$$\Rightarrow x=k\pi+\frac{\pi}{4}=\begin{cases}\text{local maximum} & \text{if } \ k \text{ even}\\ \text{local minimum} & \text{if } \ k \text{ odd}\end{cases}$$