Chain Rule

We know that for the function

$f(x) = Sin\, x\,, \qquad f'(x) = Cos\, x\,.$ **Also if**

$g(x) = 4x\,,$ **then** $g'(x) = 4.$

Now if we consider the composite function

$y = f(g(x)),$ **then we get**

$y = f(4x) = Sin(4x)$

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$y = Sin(4x)$ **What is** $y'?$

The Chain Rule will give us the answer, indeed

$$y = f(g(x)) \Rightarrow y' = f'(g(x)) \times g'(x)$$

So for this example, $f(x) = Sin\, x, \qquad g(x) = 4x$

$$f'(x) = Cos\, x, \qquad \Rightarrow f'(g(x)) = Cos\, 4x,$$

$$g(x) = 4x \Rightarrow g'(x) = 4.$$

$$\Rightarrow y' = Cos(4x) \times 4 \quad = 4Cos(4x)$$

$$y = Sin(4x)$$

$$y' = Cos(4x) \times \qquad 4$$

More Exercise: Find y', if

$a)\ y = Cos(x^3)$ $\qquad$ $b)\ y = Cos^3(x)$

$c)\ y = (3x^5 - 9x^2 + 5x - 8)^{10}$

$d)\ y = \sqrt[4]{x^4 + 2x + 1}$

$e)\ y = Cot^6(2x^5 - 3x^2)$ $\qquad$ $f)\ \dfrac{d}{dx}\tan^9(\sin x)\Big|_{x=0}$

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$$a)\ y = Cos(x^3)$$

$$y' = -Sin(x^3) \times 3x^{3-1}$$

$$b)\ y = Cos^{3}(x)$$

$$y = (Cos\ x)^{3} \qquad x^{3} \longrightarrow 3x^{3-1}$$

$$y' = 3(Cos\ x)^{3-1} \times (-Sin\ x)$$

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$$c)\ y = (3x^5 - 9x^2 + 5x - 8)^{10}$$

$$x^{10} \longrightarrow 10x^{10-1}$$

$$y' = 10(3x^5 - 9x^2 + 5x - 8)^{10-1}$$

$$\times\ (3 \times 5 \times x^{5-1} - 9 \times 2x^{2-1} + 5)$$

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$$d)\ y = \sqrt[4]{x^4 + 2x + 1}$$

$$y = \left(x^4 + 2x + 1\right)^{\frac{1}{4}}$$

$$y' = \frac{1}{4}\left(x^4 + 2x + 1\right)^{\frac{1}{4}-1} \times \left(4x^{4-1} + 2\right)$$

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$$e)\, y = Cot^{6}(2x^{5} - 3x^{2})$$

$$6\cot^{6-1}(2x^{5} - 3x^{2}) \times \left(-\csc^{2}(2x^{5} - 3x^{2})\right) \times$$

$$(2 \times 5x^{5-1} - 3 \times 2x^{2-1})$$

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$$f)\ \frac{d}{dx}\tan^9(\sin x)\Big|_{x=0}$$

$$y = \tan^9(\sin x)$$

$$y' = 9\tan^{9-1}(\sin x)\times \sec^2(\sin x)\times (\cos x)$$

$$\Rightarrow \frac{d}{dx}\tan^9(\sin x)\Big|_{x=0} = 9\tan^{9-1}(\sin 0)\times\sec^2(\sin 0)\times\cos 0$$

$$= 9\tan^{9-1}(0)\times\sec^2(0)\times 1 = 9\times 0\times 1\times 1 = 0$$

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Another Notation of Chain Rule

If y is terms of a variable t and t is terms of x

then evidently y is terms of x

so we can talk about the derivative

of y with respect to x, that is, $\frac{dy}{dx}$ and

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

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Exercise. $y = t - t^3, \quad x = t + t^3.$ $\left.\dfrac{dy}{dx}\right|_{t=1} = ?$

Solution.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dt} = 1 - 3t^2 \qquad \frac{dt}{dx} = \frac{1}{\dfrac{dx}{dt}} \quad \Rightarrow \frac{dt}{dx} = \frac{1}{1+3t^2}$$

$$\Rightarrow \frac{dy}{dx} = (1-3t^2) \times \frac{1}{1+3t^2} = \frac{1-3t^2}{1+3t^2}$$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{1-3}{1+3} = -\frac{2}{4} = -\frac{1}{2}$$

Exericse. $y = t - t^3, \quad x = t + t^3. \qquad \left.\frac{d^2y}{dx^2}\right|_{t=1} = ?$

$$\frac{dy}{dx} = \frac{1-3t^2}{1+3t^2} \qquad \frac{d^2y}{dx^2} = \frac{d\left(\frac{1-3t^2}{1+3t^2}\right)}{dx}$$

$$= \frac{d\left(\frac{1-3t^2}{1+3t^2}\right)}{dt} \times \frac{dt}{dx}$$

$$= \frac{-3\times 2t\times(1+3t^2) - 3\times 2t\times(1-3t^2)}{\left(1+3t^2\right)^2} \times \frac{1}{1+3t^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} =$$

$$\frac{-3\times 2t\times(1+3t^2)-3\times 2t\times(1-3t^2)}{\left(1+3t^2\right)^2}\times\frac{1}{1+3t^2}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{-6\times 4-6\times(-2)}{16}\times\frac{1}{4}$$

$$= \frac{-12}{16}\times\frac{1}{4} = \frac{-3}{16}$$

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More General Form of Chain Rule

$$\frac{dy}{dx_n} = \frac{dy}{dx_1} \times \frac{dx_1}{dx_2} \times \frac{dx_2}{dx_3} \times \frac{dx_3}{dx_4} \times \ldots \times \frac{dx_{n-1}}{dx_n}$$

Exercise.

$$y \sin z + y^3 = z^2 + 1$$

$$(z+2)^3 = x^3 + z(x-2) \qquad x^5 t^3 = x(t-1) + 32$$

$$\left.\frac{dy}{dt}\right|_{t=1} = ?$$